The classification of categories generated by an object of small dimension

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Declaration

The work in this thesis is my own except where otherwise stated.

The material in Chapter 3 is included in the preprint [13] titled "The Brauer-Picard groups of the ADE fusion categories". This paper is to appear in the *International Journal of Mathematics*.

Cain Edie-Michell

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Abstract

The goal of this thesis is to attempt the classification of unitary fusion categories generated by a normal object (an object comuting with its dual) of dimension less than 2. This classification has recently become accessible due to a result of Morrison and Snyder, which shows that any such category must be a cyclic extension of an adjoint subcategory of one of the ADE fusion categories. Our main tool is the classification of graded categories from [17], which classifies graded extensions of a fusion category in terms of the Brauer-Picard group, and Drinfeld centre of that category.

We compute the Drinfeld centres, and Brauer-Picard groups of the adjoint subcategories of the ADE fusion categories. Using this information we apply the machinery of graded extensions to classify the cyclic extensions that are generated by a normal object of dimension less than 2, of the adjoint subcategories of the ADE fusion categories. Unfortunately, our classification has a gap when the dimension of the object is $\sqrt{2+\sqrt{2}}$ corresponding to the possible existence of an interesting new fusion category. Interestingly we prove the existence of a new category, generated by a normal object of dimension $2\cos(\frac{\pi}{18})$, which we call the DEE fusion category. We include the fusion rules for the DEE fusion categories in an appendix to this thesis.

Contents

Chapter 1

Introduction

Unitary fusion categories are an important class of algebraic objects, providing a unifying framework for operator algebras, representation theory, and physics. In the area of operator algebras, unitary fusion categories appear as the even part of the standard invariant of a subfactor. Using the machinery available to unitary fusion categories, many strong results regarding subfactors have been obtained. For example, the fact that the index of a subfactor must be a cyclotomic integer, follows directly from an analogous result regarding dimensions of objects in fusion categories. In physics, the value of a point in a fully extended (2+1) dimensional topological quantum field theory is a unitary fusion category. In fact there is an exact correspondence between (2+1) dimensional TQFT's, and unitary fusion categories [11]. For representation theory, unitary fusion categories arise as the representation category of many algebraic objects such as quantum groups and vertex operator algebras. This unifying framework has provided deep connections between these three areas of math, for instance a Turaev-Viro TQFT can be constructed from each quantum group.

Fusion categories can be thought of as a generalisation of the representation category of a finite group, where we allow the tensor product to be non-commutative. Inspired by the classification of finite simple groups, research on fusion categories is focused towards providing classification theorems. However a complete classification is hopelessly out of reach with our current techniques, so instead research tends to focus on partial classifications.

There are many examples of such partial classifications in the literature, with each taking a different approach on what they mean by partial. For example, in [10], Deligne gives a classification of symmetric fusion categories (fusion categories with a braiding satisfying an additional symmetry relation). He is able to prove that any symmetric fusion category is equivalent to the representation category of a finite super group. In another direction, the results of [46] give a classification of pivotal fusion categories with three or fewer simple objects. While both of these results have been very influential to the field, they unfortunately failed to produce new exotic examples of fusion categories (such as was done for finite simple groups in their classification). In the former case, the classification reduced to group theory, while in the latter, everything was related to quantum groups.

Another direction of partial classification has been to consider categories generated by an object of small dimension. This approach has proven successful in providing exotic new examples of fusion categories. The extended Haagerup subfactor [3] was discovered through this type of partial classification (see [33] for an overview). The fusion categories associated to this subfactor remain one of the few examples of fusion categories with no known connection to finite or quantum groups. The only other examples of fusion categories not related to finite or quantum groups are Izumi's quadratic categories, which are close analogues of the Haagerup subfactor [26].

One of the earliest results in the field (in fact pre-dating the definition of fusion categories by several years) is the classification of unitary fusion categories generated by a self-dual object of dimension less than 2. Initially proved in the language of subfactors in the papers [4, 28, 29, 32, 35, 50, 45], such a category must be one of the $ADET$ unitary fusion categories. These are the fusion categories whose fusion graph for tensoring with the generating object of dimension less than 2, is one of the $ADET$ Dynkin diagrams, A_N, D_{2N}, E_6, E_8, or T_N. Since their discovery through this classification, these fusion categories have been extensively studied in many different contexts, and still remain some of the most important examples of unitary fusion categories.

With the above $ADET$ classification in mind, it is natural to attempt to drop the condition that the generating object X be self-dual, and obtain a full classification of unitary fusion categories generated by an object of dimension less than 2. If we assume that the category is braided, then such a classification was achieved in [18]. Here they find that such a category must be closely related to the A_N fusion categories, and thus no new exotic categories appeared in this classification. To attempt the complete classification one might emulate the techniques used in the self-dual case. However straightaway one arrives at the stumbling block that there is no classification of directed graphs of norm less than 2. Thus without a braiding assumption a classification result still appears out of reach with the current tools available. However the following Theorem of Morrison and Snyder may allow us to give a partial result. While this proof has not yet appeared in the literature, we give a sketch of the proof in the Preliminaries.

Theorem 1.0.1. Let C be a unitary fusion category generated by a normal object X of dimension less than 2, then C is a unitary cyclic extension of the adjoint subcategory of an ADE fusion category.

Given a fusion category C the results of [17] allow us to classify G-graded extensions of C. This main ingredient of such a classification is $\mathrm{BrPic}(C)$, the group of invertible bimodules over C. As well as being useful in classification problems this group also appears in the study of subfactors. If C is unitary then $\mathrm{BrPic}(C)$ classifies all subfactors whose even and dual even parts are both C. The process of computing Brauer-Picard groups of fusion categories is currently receiving attention in the literature by both researchers interested in subfactors [24, 25], and fusion categories [44, 39, 7].

With Theorem 1.0.1 in hand, along with the classification of graded extensions of [17], it now becomes feasible to attempt to classify unitary fusion categories generated by a normal object X of dimension less than 2. There is of course significant work to be done to complete such a classification. Namely we need to compute the Brauer-Picard groups of the adjoint subcategories of the ADE fusion categories, which is no easy task for even single examples, let alone for infinite families of fusion categories. Further, we have to understand and apply the classification results of [17] to classify cyclic extensions of the adjoint subcategories of the ADE fusion categories. Again this is no easy task, in fact entire papers [23] have dealt with constructing cyclic extensions of certain fusion categories.

The purpose of this thesis is to attempt the classification of unitary fusion categories generated by a normal object X of dimension less than 2. Our approach towards providing this classification will follow the proposal outlined above. While we are able to make significant progress, our classification contains a gap when the dimension of X is $\sqrt{2+\sqrt{2}}$. We discuss this gap, along with our ideas to fill it at the end of Chapter 4. This thesis will culminate with proving the following Theorem.

Theorem 1.0.2. Let C be a unitary fusion category generated by a normal object X of dimension less than 2, but not equal to $\sqrt{2+\sqrt{2}}$. Then the dimension of X is equal to $2\cos(\frac{\pi}{N+1})$ for $N \in \mathbb{N}/\{7\}$, and

- N is even, in which case C is monoidally equivalent to one of

$$\mathrm{Ad}(A_N) \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

for $M \in \mathbb{N}$, or

- $N \pmod 4 \equiv 3$ and $N \neq \{3, 7, 11\}$ in which case C is monoidally equivalent to one of

  $$\langle (f^{(1)}, 1) \rangle \subset A_N \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z})$$

  or

  $$\langle (f^{(1)}, 1) \rangle \subset A_N \overset{\omega}{\rtimes} \mathbb{Z}/M\mathbb{Z}$$

  for $M \in 2\mathbb{N}$, or

- $N \pmod 4 \equiv 1$ and $N \neq \{5, 17, 29\}$, in which case C is monoidally equivalent to one of

  $$\langle (f^{(1)}, 1) \rangle \subset A_N \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$
  $$\langle (f^{(1)}, 1) \rangle \subset A_N \overset{\omega}{\rtimes} \mathbb{Z}/M\mathbb{Z},$$

  or

  $$\langle (f^{(1)}, 1) \rangle \subset D^{\pm}_{\frac{N+3}{2}} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z})$$

  for $M \in 2\mathbb{N}$, or

- $N = 3$, in which case C is monoidally equivalent to one of

  $$\langle (f^{(1)}, 1) \rangle \subset A_3 \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z})$$

  for $M \in 2\mathbb{N}$, or

  $$\mathrm{GMR}_{\mathbb{Z}/2\mathbb{Z}}(M, \omega)$$

  for $M \in 4\mathbb{N}$, or

- $N = 5$, in which case C is monoidally equivalent to one of

  $$\langle (f^{(1)}, 1) \rangle \subset A_5 \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$
  $$\langle (f^{(1)}, 1) \rangle \subset A_5 \overset{\omega}{\rtimes} \mathbb{Z}/M\mathbb{Z},$$
  $$\langle (f^{(1)}, 1) \rangle \subset D^{\pm}_4 \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z})$$

  for $M \in 2\mathbb{N}$, or

  $$\mathrm{GMR}^{T}_{\mathbb{Z}/3\mathbb{Z}}(M, \omega, \pm),$$

  or

  $$\mathrm{GMR}^{V}_{\mathbb{Z}/3\mathbb{Z}}(M, \omega, \pm),$$

  for $M \in 6\mathbb{N}$, or

- $N = 11$, in which case C is monoidally equivalent to one of

$$\langle (f^{(1)}, 1)\rangle \subset A_{11} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

$$\langle (f^{(1)}, 1)\rangle \subset A_{11} \overset{\omega}{\rtimes} \mathbb{Z}/M\mathbb{Z},$$

$$\langle (f^{(1)}, 1)\rangle \subset E_6^{\pm} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

or

$$\langle (f^{(1)}, 1)\rangle \subset E_6^{\pm} \overset{\omega}{\rtimes} \mathbb{Z}/M\mathbb{Z},$$

for $M \in 2\mathbb{N}$, or

- $N = 17$, in which case C is monoidally equivalent to one of

$$\langle (f^{(1)}, 1)\rangle \subset A_{17} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

$$\langle (f^{(1)}, 1)\rangle \subset A_{17} \overset{\omega}{\rtimes} \mathbb{Z}/M\mathbb{Z},$$

$$\langle (f^{(1)}, 1)\rangle \subset D_{10}^{\pm} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

for $M \in 2\mathbb{N}$, or

$$\langle (\Omega, 1)\rangle \subset DEE^{\pm}(\psi) \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

for $\psi \in H^3(\mathbb{Z}/6\mathbb{Z}, \mathbb{C}^{\times})$ and $M \in 6\mathbb{N}$, or

- $N = 29$, in which case C is monoidally equivalent to one of

$$\langle (f^{(1)}, 1)\rangle \subset A_{29} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

$$\langle (f^{(1)}, 1)\rangle \subset A_{29} \overset{\omega}{\rtimes} \mathbb{Z}/M\mathbb{Z},$$

$$\langle (f^{(1)}, 1)\rangle \subset D_{16}^{\pm} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

or

$$\langle (f^{(1)}, 1)\rangle \subset E_8^{\pm} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

for $M \in 2\mathbb{N}$.

In each case, $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^{\times})$.

In this Theorem, $f^{(1)}$ is the standard small dimensional generating object each of the ADE fusion categories. Details on the ADE categories can be found in Chapter 2. Details including fusion rules for the fusion categories $\mathrm{GMR}_{\mathbb{Z}/2\mathbb{Z}}$, $\mathrm{GMR}^{T}_{\mathbb{Z}/3\mathbb{Z}}$, and $\mathrm{GMR}^{V}_{\mathbb{Z}/3\mathbb{Z}}$ can be found in Chapter 4. Details on the DEE categories can be found in Chapter 4, and fusion rules are given in Appendix A.

We begin our thesis with the preliminaries in Chapter 2. Here we define many concepts relating to the topic of this thesis, including but not limited to, the definition of a fusion category, bimodules categories, the Brauer-Picard group, the classification of graded categories, and planar algebras. In particular we define the ADE planar algebras, and associated unitary fusion categories. These categories will play a key role in this thesis, due to Theorem 1.0.1.

While the $ADET$ classification of unitary fusion categories generated by a self-dual object of dimension less than 2 has long been known as a corollary of the classification of subfactors of index less than 2, we include a new version of the proof, completely independent from the subfactor classification. Part of the motivation for including this proof is that we could not find the statement of the classification anywhere in the literature, despite the result being so well known. We also point out that our proof counts exactly how many unitary fusion categories there are of each type. That is there are exactly two unitary fusion categories of type A_N, four of type D_{2N}, four of type E_6, four of type E_8, and just one of type T_N.

We conclude the preliminaries by communicating a proof of Theorem 1.0.1.

In Chapter 3 we compute the Brauer-Picard groups of the adjoint subcategories of the ADE fusion categories. This computation makes use of the fact that the Brauer-Picard group of a category is isomorphic to the group of braided auto-equivalences of the Drinfeld centre of that category. Thus we spend the first part of this Chapter computing the Drinfeld centres of the adjoint subcategories of the ADE fusion categories. In particular we give planar algebra presentations for these categories.

Using these planar algebra presentations we then compute the braided planar algebra automorphisms. With the help of certain planar algebra machinery, these braided planar algebra automorphisms give us braided auto-equivalences of the associated Drinfeld centres. While these do not give us all the braided auto-equivalences, we find that they give us enough that we can use combinatorics and group theory to completely determine the entire group of braided auto-equivalences of the centre, and hence the Brauer-Picard groups of the adjoint subcategories of the ADE fusion categories.

We find two exciting cases with interesting Brauer-Picard groups. These are the unitary fusion categories $\mathrm{Ad}(A_7)$ and $\mathrm{Ad}(D_{10})$. These categories have Brauer-Picard groups $D_{2\cdot 4}$ and $S_3 \times S_3$ respectively. This is important to us because the existence of interesting bimodules over these categories means the possible existence of interesting cyclic extensions, and thus possible exotic categories appearing in our classification of unitary fusion categories generated by a normal object of dimension less than 2. In fact the DEE fusion categories, along with the gap at $\sqrt{2+\sqrt{2}}$ appear in our classification for exactly this reason. We end the Chapter by developing combinatorial arguments to explicitly describe the bimodules over each of these two cases, as this information will help us when trying to classify cyclic extensions of these categories.

In Chapter 4 we complete the proof of Theorem 1.0.2 by classifying cyclic extensions of the adjoint subcategories of the ADE fusion categories, generated by an object of dimension less than 2. We begin by summarising the results of the previous Chapter, needed to apply the results of [17] to classify such extensions. This includes explicit realisations of all the invertible bimodules over the adjoint subcategories of the ADE fusion categories, along with the dimensions of the objects in these categories, and the invertible objects in the Drinfeld centres.

Working through the extension theory on a case by case basis we arrive at a proof of Theorem 1.0.2. For most of the examples we are able to explicitly construct all the possible cyclic extensions that we are interested in, and not have to worry about the intricacies of the extension theory. However there are several cases that prove more difficult. These are the categories $\mathrm{Ad}(A_3)$, $\mathrm{Ad}(D_4)$, and $\mathrm{Ad}(D_{10})$. In each of these cases we have to use high powered machinery from the extension theory of graded extensions to give the classification. These proofs unfortunately end up being quite messy. Even worse, in the $\mathrm{Ad}(A_7)$ case we are not able to classify all cyclic extensions generated by an object of dimension less than 2, hence the gap in our classification at dimension $\sqrt{2+\sqrt{2}}$.

Chapter 2

Preliminaries

Fusion Categories, and Additional Structure and Properties

We begin by defining a fusion category. As we do not give a large amount of detail, we point the reader towards [16] for a more in depth definition.

Definition 2.0.1. [16] Let $\mathbb{K}$ be an algebraically closed field. A fusion category over $\mathbb{K}$ is a rigid semisimple $\mathbb{K}$-linear monoidal category C with finitely many isomorphism classes of simple objects and finite dimensional spaces of morphisms, such that the unit object $\mathbf{1}$ of C is simple.

For the purpose of this thesis our algebraically closed field will be $\mathbb{C}$. To avoid repetition whenever we refer to a fusion category from now on we implicitly mean a fusion category over $\mathbb{C}$.

Example 2.0.2. Let G be a finite group, then the category of finite dimensional complex representations of G is a fusion category.

We say a fusion category is a C^*-fusion category if it comes with the additional structure of an involutive anti-linear contravariant endofunctor $*$, such that the hom spaces are Banach spaces with respect to the norm induced by $*$, and the induced norm must satisfy the conditions

$$||f \circ g|| \leq ||f|| ||g|| \text{ and } ||f^* \circ f|| = ||f||^2.$$

We say a fusion category is unitary if it is C^*, and additionally, all structure isomorphisms (such as associators) are unitary. All of the examples of fusion categories we study in this Thesis are unitary.

A pivotal fusion category is a fusion category along with an isomorphism from the double dual functor to the identity functor. Any unitary fusion category has a canonical pivotal structure.

In a pivotal fusion category C, one can define the dimension of an object. This definition is a generalisation of the dimension of a representation of a finite group. Let $X \in C$, then $\dim(X)$ is defined as the scalar

$$\mathrm{ev}_{X^*} \circ (\mathrm{pv}_X \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X : \mathbf{1} \to \mathbf{1}.$$

This dimension function is additive with respect to direct sums, and multiplicative with respect to tensor products. When C is unitary, every object has positive dimension under the canonical pivotal structure.

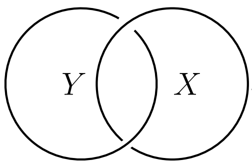
Given a fusion category C, we define the global dimension of C as

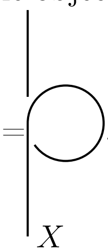
$$\dim(C) := \sum_{\mathrm{Irr}(C)} \dim(X)^2,$$

where $\mathrm{Irr}(C)$ is the set of simple objects of C, up to isomorphism.

A braided fusion category is a fusion category along with a natural isomorphism called the braiding $\gamma_{X,Y} : X \otimes Y \cong Y \otimes X$ that satisfies certain naturality conditions. For details see [34].

There are two useful invariants one can compute for a pivotal braided fusion category C. They are the S and T matrices, indexed by simple objects of C.

$S_{X,Y} =$ 

, $\qquad T_{X,X} =$ 

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If a pivotal braided tensor category has invertible S-matrix then we say that the category is modular. An important example of a modular category is the Drinfeld centre of a spherical fusion category C.

Definition 2.0.3. The Drinfeld centre of C is the modular tensor category $Z(C)$ whose:

- Objects are pairs (X, γ) where $X \in C$ and γ is a natural isomorphism

  $$\gamma : X \otimes ? \to ? \otimes X$$

  such that for all $Y \in C$ we have

  $$\gamma_{Y \otimes Z} = (\mathrm{id}_Y \otimes \gamma_Z) \circ (\gamma_Y \otimes \mathrm{id}_Z).$$

- Morphisms are

  $\mathrm{Hom}((X,\gamma),(Y,\lambda)) := \{f \in \mathrm{Hom}_C(X,Y) : (\mathrm{id}_Z \otimes f) \circ \gamma_Z = \lambda_Z \circ (f \otimes \mathrm{id}_Z) \text{ for all } Z \in C\}.$

- Tensor product is

$$(X,\gamma)\otimes(Y,\lambda) := (X\otimes Y, (\gamma\otimes \mathrm{id}_Y)\circ(\mathrm{id}_X\otimes\lambda)).$$

- Braiding is

$$\gamma_{(X,\gamma)\otimes(Y,\lambda)} := \lambda_X.$$

For a given fusion category C the collection of monoidal auto-equivalences of C forms a monoidal category, with the objects being the monoidal auto-equivalences, and the morphisms being the monoidal natural isomorphisms. We call this monoidal category $\underline{\mathrm{Aut}_\otimes}(C)$. Similarly for C a braided category, we write $\underline{\mathrm{Aut}^{\mathrm{br}}}(C)$ for the monoidal category of braided auto-equivalences. We write $\mathrm{Aut}_\otimes(C)$ to mean the group of tensor auto-equivalences of C, up to natural isomorphism. Similarly we write $\mathrm{Aut}^{\mathrm{br}}(C)$ to mean the group of braided auto-equivalences, up to natural isomorphism.

Graded Categories

Let C be a fusion category, and G a finite group. We say C is G-graded if

$$C = \bigoplus C_g$$

for C_g non-trivial abelian subcategories of C, such that the tensor product restricted to $C_g \times C_h$ has image in C_{gh}.

We say C is a G-graded extension of C_e if C is graded, with trivial component C_e.

For any fusion category C, we define the adjoint subcategory of C as

$$\mathrm{Ad}(C) := \langle X\otimes X^* : X \in C\rangle,$$

where $\langle . \rangle$ denotes the full subcategory generated by all subobjects. The category C is a graded extension of $\mathrm{Ad}(C)$.

For a fixed fusion category C, there is a classification of graded extensions of C. We describe the details of this classification later in this Chapter.

Crossed product fusion categories

Important examples of a graded categories are the crossed product fusion categories, first defined in [19]. To define a crossed product fusion category, we first need to define categorical 1-groups.

Definition 2.0.4. Let G be a finite group. We define the monoidal category $\underline{G}$ as the monoidal category whose objects are the elements of G, and morphisms are identities. The tensor product is the multiplication in G, and the associator is trivial.

Definition 2.0.5. Let C be a fusion category, G a finite group, $\omega \in H^3(G, \mathbb{C}^\times)$, and $\Gamma : \underline{G} \to \underline{\mathrm{Aut}_\otimes}(C)$ a monoidal functor. We define $C \overset{\omega}{\rtimes} G$ as the abelian category

$$\bigoplus_G C,$$

with tensor product

$$(X_1, g_1) \otimes (X_2, g_2) := (X_1 \otimes \Gamma(g_1)[X_2], g_1 g_2),$$

and associator

$$[(X_1, g_1) \otimes (X_2, g_2)] \otimes (X_3, g_3) \to (X_1, g_1) \otimes [(X_2, g_2) \otimes (X_3, g_3)]$$

given by the isomorphism

$$\omega_{g_1, g_2, g_3} \mathrm{id}_{X_1} \otimes \tau^{g_1}_{X_2, \Gamma(g_2)[X_3]} \circ \mathrm{id}_{X_1} \otimes \mathrm{id}_{\Gamma(g_1)[X_2]} \otimes \mu_{g1, g2}.$$

Here τ^{g_1} is the tensorator for the functor $\Gamma(g_1)$, and μ is the tensorator for the functor Γ.

Crossed product fusion categories are of interest to us as they will appear in our classification of unitary categories generated by a normal object of dimension less than 2.

Module Categories and Bimodule Categories

Here we define module categories, algebra objects, bimodules categories, and the Brauer-Picard group.

Definition 2.0.6. [45] A left module category M over a fusion category C is a semi-simple $\mathbb{C}$-linear category along with a functor $\otimes : C \times M \to M$, and natural isomorphisms $(X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$ satisfying a straightforward pentagon equation.

Example 2.0.7. The category Vec, of finite dimension vector spaces, is a module category over the fusion category $\mathsf{Rep}(G)$ for every finite group G. The action is given via the forgetful functor $\mathsf{Rep}(G) \to \mathsf{Vec}$.

Strongly related to the theory of modules over fusion categories are algebra objects in fusion categories.

Definition 2.0.8. [45] An algebra object in a fusion category is an object A together with morphisms $\mathbf{1} \to A$ and $A \otimes A \to A$ satisfying associator and unit axioms.

One can define left (right) module objects over an algebra A in C. The category of left (right) A modules in C forms a right (left) module category over C, we write $A - \text{mod}$ for this category. A result of Ostrik [45] shows that every right (left) semisimple module category over C arises as the category of left (right) A-modules for some algebra A in C. Similarly the category of bimodule objects over an algebra A forms a fusion category, we write A-bimod for this fusion category.

The following Lemma about algebra objects will be useful later in this Thesis.

Lemma 2.0.1. Let G be a finite group, and $C \simeq \bigoplus_G C_g$ be a G-graded fusion category. If $A \in C_e$ is an algebra object then

$$A - \text{bimod}_C \simeq \bigoplus_G A - \text{bimod}_{C_g},$$

where

$$A - \text{bimod}_{C_g} \coloneqq C_g \cap (A - \text{bimod}_C).$$

Proof. There is a fully faithful functor $\bigoplus_G A - \text{bimod}_{C_g}$ to $A - \text{bimod}_C$ simply given by forgetting the homogeneous grading. Each of the $A - \text{bimod}_{C_g}$ is an invertible bimodule over $A - \text{bimod}_{C_e}$, so each has global dimension equal to that of $A - \text{bimod}_{C_e}$. The global dimension of $\bigoplus_G A - \text{bimod}_{C_g}$ is therefore $|G| \dim(A - \text{bimod}_{C_e}) = |G| \dim(C_e)$. The global dimension of $A - \text{bimod}_C$ is $\dim(C) = |G| \dim(C_e)$. Thus the above functor is a fully faithful functor between categories with the same global dimensions, and hence is an equivalence by [12, Proposition 2.11]. □

A slight generalisation of a module category over C, is the notion of a bimodule category over C. This is a straightforward generalisation where now the category C can act on both the left and right, and there is the additional structure of an isomorphism relating the left and right actions (see [22] for an explicit definition).

Given a C bimodule and an auto-equivalence of C, one can construct a new bimodule. This construction will be important later in this thesis.

Definition 2.0.9. Let M be a bimodule category over C, and $F : C \to C$ a monoidal auto-equivalence. We define a new bimodule ${}_F M$, which is equal to M as a right module category, and with left action given by

$$X \triangleright_{{}_F M} m := F(X) \triangleright_M m.$$

The structure morphisms for ${}_F M$ consist of a combination of the structure morphisms for M, and the tensorator of F.

Given two bimodules (whose sources and targets correspond) we can define their relative tensor product, which is a new bimodule category.

Definition 2.0.10. [17] Let M, N be bimodule categories over C. The tensor product of M with N is a semi-simple category $M \boxtimes_C N$ together with a C-balanced functor (see [22] for a definition)

$$B_{M,N} : M \times N \to M \boxtimes_C N$$

inducing, for every semi-simple category A, an equivalence between the category of C-balanced functors from $M \times N$ to A and the category of functors from $M \boxtimes_C N$ to A:

$$\mathrm{Fun}_{\mathrm{bal}}(M \times N, A) \cong \mathrm{Fun}(M \boxtimes_C N, A).$$

Using this tensor product of bimodules we can define the Brauer-Picard group of C.

Definition 2.0.11. Let C be a fusion category. The Brauer-Picard group of C,which we denote $\mathrm{BrPic}(C)$, is the group of invertible C-bimodules with respect to the relative tensor product.

In practice the Brauer-Picard group of a fusion category is extremely hard to compute directly. Thankfully the following isomorphism of groups gives us an alternative way to compute it.

$$\mathrm{Aut}^{\mathrm{br}}(Z(C)) \cong \mathrm{BrPic}(C).$$

This isomorphism is proved in Theorem 1.1 of [17].

The classification of graded categories

Here we recall the classification of graded extensions from [17]. This classification result will play a large role in this Thesis, as we plan to classify all cyclic extensions, generated by a normal object of dimension less than 2, of the adjoint subcategories of the ADE fusion categories.

An important piece of data in the classification of graded extensions of a fusion category C is $\underline{\underline{\mathrm{BrPic}}}(C)$, the 3-group of invertible bimodules over a fusion category C. This is a categorification of Brauer-Picard group of C defined earlier. More details can be found in the papers [22, 17].

We have in $\underline{\underline{\mathrm{BrPic}}}(C)$ that there is a single 0-morphism, the category C. The 1-morphisms from $C \to C$ are the invertible C bimodules M, the 2-morphisms $M \to N$ are bimodule equivalences F, and 3-morphisms $F \to G$ are natural isomorphisms of bimodule functors μ. Composition at the 2-level is the composition $\circ$ of bimodule functors, and composition at the 3-level is the vertical composition $\cdot$ of natural transformations. Composition at the 1-level is given by the relative tensor product of bimodule categories.

Recall from [17] that G-graded extensions of C are classified by

- a group homomorphism $c : G \to \mathrm{BrPic}(C)$, such that a certain element $o_3(c) \in H^3(G, \mathrm{Inv}(Z(C)))$ is trivial,
- an element M of an $H^2(G, \mathrm{Inv}(Z(C)))$-torsor, such that a certain element $o_4(c, M) \in H^4(G, \mathrm{Inv}(Z(C)))$ is trivial,
- an element A of an $H^3(G, \mathbb{C}^\times)$-torsor.

Here the element M is a collection of bimodule equivalences $M_{g,h} : c_g \boxtimes c_h \to c_{gh}$, and the element A is a collection of bimodule natural isomorphisms

$$A_{f,g,h} : M_{fg,h}(M_{f,g} \boxtimes \mathrm{Id}_{c_h}) \to M_{f,gh}(\mathrm{Id}_{c_f} \boxtimes M_{g,h}).$$

The action of $T \in H^2(G, \mathrm{Inv}(Z(C)))$ on M is given by

$$(T \triangleright M)_{g,h} := T_{g,h} \boxtimes M_{g,h},$$

and the action of $H^3(G, \mathbb{C}^\times)$ on A is simply given by scaling. Formulas for the obstructions $o_3(c)$ and $o_4(c, T)$ can be found in [17, Section 8], and graphical descriptions can be found in [14, Section 4].

With the triple of data (c, M, A) we can reconstruct the corresponding graded extension as follows. As a plain abelian category, the extension is

$$\bigoplus_G c_g.$$

The tensor product is given by M, and the associativity isomorphisms are given by A.

We note that this classification of graded extensions is only up to equivalence of extensions, and not monoidal equivalence. An equivalence of extensions is a monoidal equivalence that is the identity on the trivially graded piece, and preserves the grading. An example of the difference between equivalence of extensions, and monoidal equivalence can be seen in the categories $\mathsf{Vec}^{\omega}(\mathbb{Z}/5\mathbb{Z})$ for $\omega \in H^3(\mathbb{Z}/5\mathbb{Z}, \mathbb{C}^\times)$, thought of as a $\mathbb{Z}/5\mathbb{Z}$ graded extension of Vec. Considered up to equivalence of extensions, every element of $H^3(\mathbb{Z}/5\mathbb{Z}, \mathbb{C}^\times)$ gives a distinct extension. However up to monoidal equivalence, there are only three distinct categories.. In practice this means that our classification of unitary categories generated by a normal object of dimension less than 2 is not up to monoidal equivalence. More detail on this over-counting of graded extensions, and attempts to remedy the situation, can be found in the author's paper [14]. In particular several examples are given that show exactly how over-counting occurs.

There is also the problem that this extension theory does not deal with unitary structures. This has been remedied in [20] where it is shown in Remark 5.16 that if a unitary fusion category C is completely unitary (as defined in the same paper), then any graded extension of C is monoidally equivalent to a unitary fusion category. Thankfully for us, as we will see later in this thesis, the adjoint subcategories of the ADE fusion categories are all completely unitary.

Planar Algebras

Planar algebras were first introduced in [31] as an axiomatisation of the standard invariant of a subfactor.

Definition 2.0.12. A planar algebra is a collection of vector spaces $\{P_n : n \in \mathbb{N}\}$ along with a multi-linear action of planar tangles.

For more details see the above mentioned paper.

Given a planar algebra such that $P_0 \cong \mathbb{C}$ one can construct a pivotal rigid monoidal category and vice versa.

Definition 2.0.13. [40] Given a planar algebra P we construct a strict monoidal category C_P as follows:

- An object of C_P is a projection in some algebra P_{2n}.
- For two projections $\pi_1 \in P_{2n}, \pi_2 \in P_{2m}$ the morphism space $\mathrm{Hom}(\pi_1, \pi_2)$ is the vector space $\pi_1 P_{n+m} \pi_2$.

- The tensor product of two projections is the disjoint union.

- The tensor identity is the empty picture.

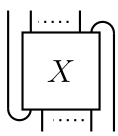
For direct sums to make sense in this constructed category one has to work in the matrix category of C_P where objects are formal direct sums of objects in C_P and morphisms are matrices of morphisms in C_P. For more details on the matrix category see [40].

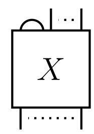
For simplicity of notation when we will write P for both the planar algebra P, and the matrix category of C_P as constructed above. The context of use should make it clear to the reader if we are referring to the planar algebra or corresponding monoidal category.

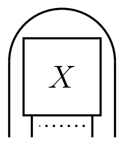
Conversely, given a pivotal rigid monoidal category C, along with choice of symmetrically self-dual object X, one can construct a planar algebra $\mathrm{PA}(C;X)_n := \mathrm{Hom}_C(\mathbf{1} \to X^{\otimes n})$. More details can be found in [27], where it is shown that this construction is inverse to the one described in Definition 2.0.13.

Some of the simplest examples of planar algebras are the ADE planar algebras. These are two infinite families A_N and D_{2N}, and two sporadic examples, E_6 and E_8. These planar algebras were given the following generator and relation presentations in [40, 2]. The ADE planar algebras (and associated fusion categories) are the main objects of study in this thesis.

To describe the ADE planar algebras, and the planar algebras in the rest of this paper, we adopt the notation of [2] for this paper so that for a diagram X:

$\rho(X) :=$ 

, $\tau(X) :=$ 

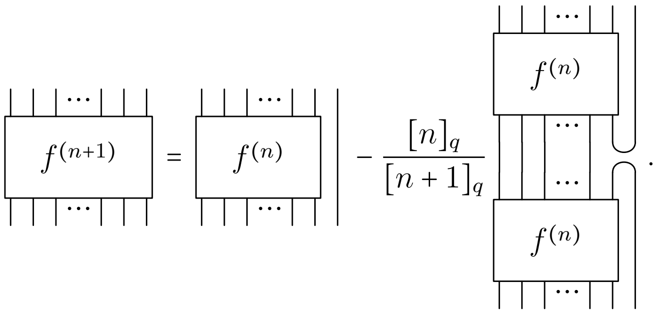
, $\hat{X} :=$ 

.

Composition of two elements in P_{2N} is given by vertical stacking with N strings pointing up and N strings pointing down.

Definition 2.0.14. Let q be a root of unity. For n a natural number we define the quantum integer $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$.

Definition 2.0.15. [49] We define the Jones-Wenzl idempotents $f^{(n)}$ in a planar algebra P via the recursive relation:

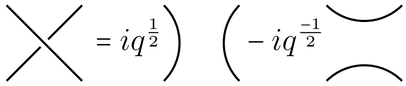


The A_N planar algebras have loop parameter $[2]_q$, no generators, and a single relation. The D_{2N}, E_6, and E_8 planar algebras have loop parameter $[2]_q$, a single uncappable generator $S \in P_k$ with rotational eigenvalue ω, satisfying relations as in the following table. Definition of all these planar algebra terms can be found in [2].

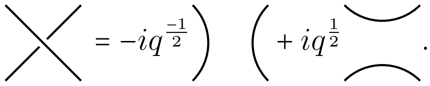
	q	k	ω	additional relations
A_N	$\pm e^{\frac{i\pi}{N+1}}$	-	-	$f^{(N)} = 0$
D_{2N}	$\pm e^{\frac{i\pi}{4N-2}}$	$4N-4$	$\pm i$	$S \otimes S = f^{(2N-2)}$ $f^{(4N-3)} = 0$
E_6	$\pm e^{\pm\frac{i\pi}{12}}$	6	q^{16}	$S^2 = S + [2]_q^2[3]_q f^{(3)}$ $\hat{S} \circ f^{(8)} = 0$
E_8	$\pm e^{\pm\frac{i\pi}{30}}$	10	q^{36}	$S^2 = S + [2]_q^2[3]_q f^{(5)}$ $\hat{S} \circ f^{(12)} = 0.$

Note that there are two distinct A_N planar algebras, and four for each of the other Dynkin diagrams. We give distinguished names to some of these different planar algebras. We will write A_N for the planar algebra with $q = e^{\frac{i\pi}{N+1}}$. We write D_{2N}^+ for the planar algebra with $q = e^{\frac{i\pi}{4N-2}}$ and $\omega = i$, and D_{2N}^- for the planar algebra with $q = e^{\frac{i\pi}{4N-2}}$ and $\omega = -i$. We write E_6^+ for the planar algebra with $q = e^{\frac{i\pi}{12}}$, and E_6^- for the fusion category with $q = e^{\frac{-i\pi}{12}}$. In a similar fashion we write E_8^+ for the planar algebra with $q = e^{\frac{i\pi}{30}}$, and E_8^- for the fusion category with $q = e^{\frac{-i\pi}{30}}$.

Remark 2.0.2. The A_N planar algebras can be equipped with a braiding. There are multiple different braidings one can choose. The two we will use in this paper are the standard braiding



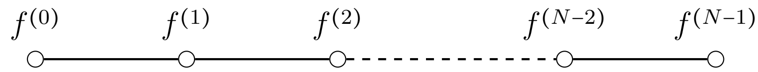
and the opposite braiding



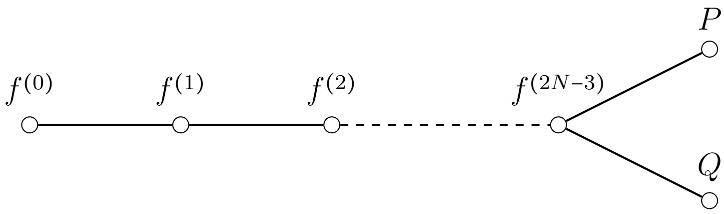
We call these braided planar algebras A_N and A_N^{bop} respectively.

From these (braided) planar algebras we get (braided) fusion categories via the idempotent completion construction described earlier. In fact these categories are unitary. We call the resulting unitary fusion categories, the ADE fusion categories. We present the simple objects of these categories in graph form. The vertices are the simple objects of the fusion category and the number edges between two simple objects X and Y counts the number of copies of Y in X tensored with the single strand $f^{(1)} \in P_2$. Presenting this information in graph form makes clear our choice of naming convention.

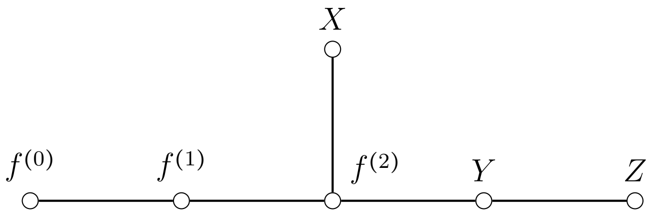
A_N :



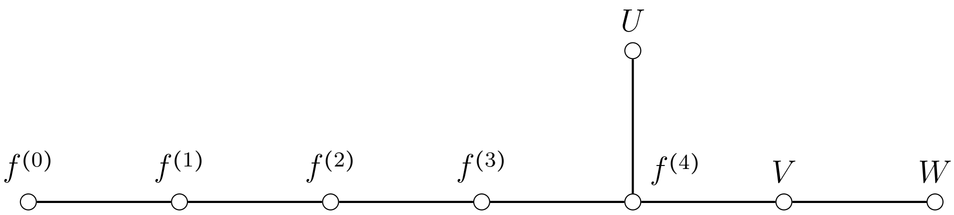
$D_{2N}^{\pm}$:



$E_6^{\pm}$:



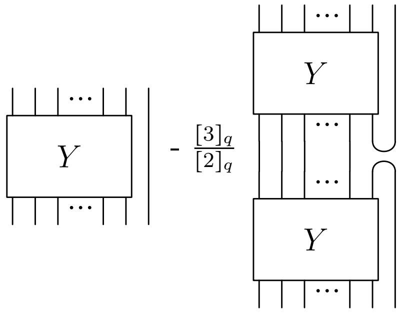
$E_8^{\pm}$:



Where:

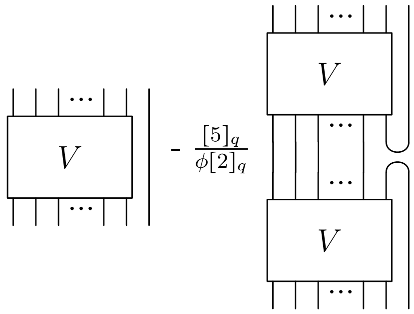
- $P := \frac{f^{(2N-2)}+S}{2}$,
  $Q := \frac{f^{(2N-2)}-S}{2}$ in D_{2N}

- $X := \frac{1}{\sqrt{3}} f^{(3)} + \left(1 - \frac{2}{\sqrt{3}}\right) S$,
  $Y := \left(1 - \frac{1}{\sqrt{3}}\right) f^{(3)} + \left(-1 + \frac{2}{\sqrt{3}}\right) S$,

  $Z :=$ 

, in E_6

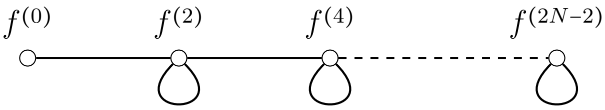
- $U := \frac{1}{4}\left(-\sqrt{5} + \sqrt{6\left(1 + \frac{1}{\sqrt{5}}\right)} + 1\right) f^{(5)} + \frac{1}{2}\left(\sqrt{5} - \sqrt{6\left(1 + \frac{1}{\sqrt{5}}\right)} + 1\right) S$,
  $V := \frac{1}{4}\left(\sqrt{5} - \sqrt{6\left(1 + \frac{1}{\sqrt{5}}\right)} + 3\right) f^{(5)} + \frac{1}{2}\left(-\sqrt{5} + \sqrt{6\left(1 + \frac{1}{\sqrt{5}}\right)} - 1\right) S$,

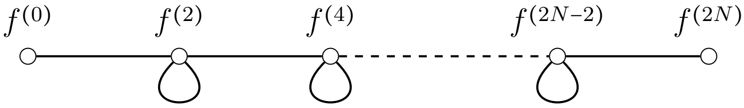
  $W :=$ 

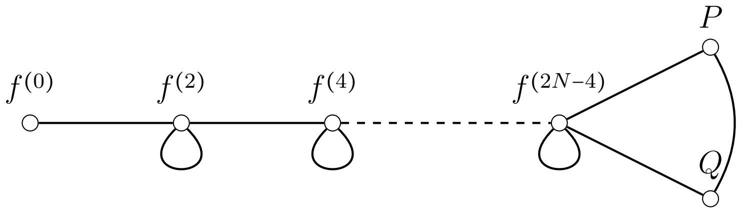
, in E_8.

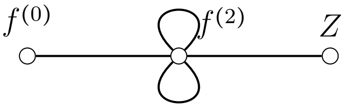
Each of these categories is $\mathbb{Z}/2\mathbb{Z}$-graded. The trivially graded piece of this grading is also a fusion category. These subcategories are the adjoint subcategories. While there were two A_N unitary fusion categories, these both have the same adjoint subcategory which we call $\mathrm{Ad}(A_N)$. Similarly while there were four D_{2N} unitary fusion categories, these all have the same adjoint subcategory, which we simply call $\mathrm{Ad}(D_{2N})$. The situation is slightly different for the E_6 and E_8 cases. The adjoint subcategories of E_6^+ and E_6^- are monoidally inequivalent, we write $\mathrm{Ad}(E_6^+)$ and $\mathrm{Ad}(E_6^-)$ for these two fusion categories. Similarly the adjoint subcategories of $\mathrm{Ad}(E_8^+)$ and $\mathrm{Ad}(E_8^-)$ are monoidally inequivalent, we write $\mathrm{Ad}(E_8^+)$ and $\mathrm{Ad}(E_8^-)$ for these two fusion categories.

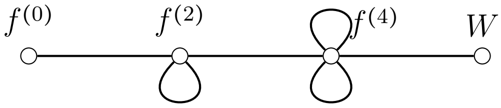
The fusion graphs for the adjoint subcategories of the ADE fusion categories are as follows:

$\mathrm{Ad}(A_{2N})$: 

$\mathrm{Ad}(A_{2N+1})$: 

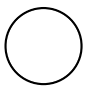
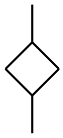
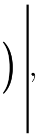
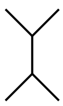
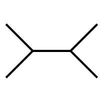
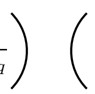
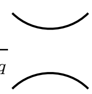
$\mathrm{Ad}(D_{2N})$: 

$\mathrm{Ad}(E_6^{\pm})$: 

$\mathrm{Ad}(E_8^{\pm})$: 

Planar algebra presentations for these adjoint subcategories can be acquired by taking the sub-planar algebra generated by the strand $f^{(2)}$ inside the full planar algebra. We explicitly describe the planar algebras for $\mathrm{Ad}(A_{2N})$ and $\mathrm{Ad}(D_{2N})$ as we will need them later on in this Thesis.

The $\mathrm{Ad}(A_N)$ planar algebra has a single generator $T \in P_3$ (which we draw as a trivalent vertex) with the following relations (with $q = e^{\frac{\pi i}{N+1}}$):

1.  $= [3]_q$,
2. $\rho(T) = T$,
3. $\tau(T) = 0$,
4.  $= \left(\frac{[3]_q - 1}{[2]_q}\right)$ ,
5.  $-$  $= \frac{1}{[2]_q}$  $- \frac{1}{[2]_q}$  .

The $\mathrm{Ad}(D_{2N})$ planar algebra has two generators $T \in P_3$ (again drawn as a trivalent vertex), and $S \in P_{2N-2}$ with all the relations of the $\mathrm{Ad}(A_{4N-3})$ planar algebra, along with :

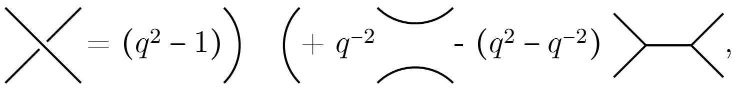
1. $\rho(S) = -S$,

2. $\tau(S) = 0$,

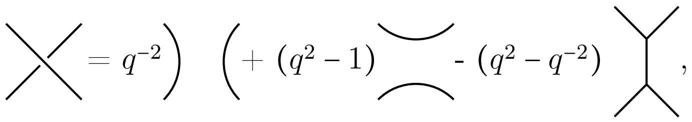
3. $S \circ (T \otimes \mathrm{Id}_{N-3}) = 0$,

4. $S \otimes S = f^{(N-1)}$.

As the $\mathrm{Ad}(A_N)$ planar algebras are sub-planar algebras of the A_N planar algebras they inherit the braiding, so can also be thought of as braided planar algebras. We can now also equip the $\mathrm{Ad}(D_{2N})$ planar algebras with a braiding. The two braidings on $\mathrm{Ad}(D_{2N})$ that we care about are



and



We call these braided planar algebras $\mathrm{Ad}(D_{2N})$ and $\mathrm{Ad}(D_{2N})^{\mathrm{bop}}$.

Note that there is some ambiguity in how we think of the objects of the adjoint subcategory. In the above fusion graphs the simple objects come directly from the corresponding full category, so the generating object for each of these is $f^{(2)}$. Yet if we construct the fusion categories from the above planar algebras the generating object will now be called $f^{(1)}$. To fix a convention for the rest of this thesis we will regard objects of the adjoint subcategory as objects of the full category.

The following theorem regarding the ADE and $\mathrm{Ad}(A_{2N})$ planar algebras is well known, however the author was unable to find a proof in the literature. For completeness we include our own version of the proof.

Theorem 2.0.3. Let C be a unitary fusion category generated by a symmetrically self-dual object X of dimension less than 2. Then C is the idempotent completion of one of the A_N, $\mathrm{Ad}(A_{2N})$, D_{2N}, E_6, or E_8 planar algebras.

Proof. As the dimension of X is less than 2, it must be equal to $q + q^{-1}$ for q some primitive N-th root of unity. Furthermore as C is unitary we must have that $q = \pm e^{\frac{i\pi}{N+1}}$.

Consider the planar algebra $\mathrm{PA}(C; X)$, as defined earlier. Due to the pivotality of C, this planar algebra contains a copy of the A_N planar algebra, with choice of q as above. Thus there is a planar algebra injection

$$A_N \to \mathrm{PA}(C; X).$$

As X tensor generates C, we can apply [27, Theorem A] to get a dominant tensor functor

$$A_N \to C,$$

now thinking of A_N as a fusion category. Thus from [8, Cor 5.6] there exists a commutative central algebra (an algebra object along with lift to the centre, such that the corresponding algebra in the centre is commutative) $(A, \sigma) \in A_N$ such that

$$C \simeq \mathrm{mod}_{A_N}(A, \sigma).$$

Here we adopt the notation of [8].

The commutative algebras in the categories A_N have been computed in [45] (in the language of quantum groups). They are

- $\mathbf{1}$ for all N, which has a unique central structure. The fusion graph for the corresponding fusion category is A_N.
- $\mathbf{1} \oplus f^{(N-1)}$ when N is even and $q = e^{\frac{i\pi}{N+1}}$, which has a unique central structure. The fusion graph for the corresponding fusion category is $T_{\frac{N}{2}}$.
- $\mathbf{1} \oplus f^{(N-1)}$ when $N \pmod 4 \equiv 1$, which has two central structures (which we call $\pm$). The fusion graph for the corresponding fusion category is $D_{\frac{N+3}{2}}$.
- $\mathbf{1} \oplus f^{(6)}$ when $N = 11$, which has two central structures. The fusion graph for the corresponding fusion category is E_6.
- $\mathbf{1} \oplus f^{(10)} \oplus f^{(18)} \oplus f^{(28)}$ when $N = 29$, which has two central structures. The fusion graph for the corresponding fusion category is E_8.

The algebra structures on each of these objects is unique.

The result now follows from a straightforward counting argument. We explicitly spell out the D_{2N} case, and leave the others to the reader.

For a fixed N, we count the number of D_{2N} planar algebras. We have two choices to make, $q = \pm e^{\frac{i\pi}{4N-2}}$, and $\pm i$ for the rotational eigenvalue of S. Thus there are 4 D_{2N} planar algebras. Each of these planar algebras can be idempotent completed to give a unitary category generated by a symmetrically self-dual object of dimension less than 2. These pivotal fusion categories have fusion graph D_{2N} for tensoring with the object of dimension less than 2.

On the other hand, if we have a pivotal fusion category generated by an symmetrically self-dual object X of dimension less than 2, with D_{2N} fusion graph for tensoring by X, then the above argument shows that this category must be

$\mathrm{mod}_{A_N}(\mathbf{1} \oplus f^{4N-4}, \pm)$ for $q = \pm e^{\frac{i\pi}{4N-2}}$. There are four such categories, thus each of these categories can be realised as the idempotent completion of a D_{2N} planar algebra. □

Remark 2.0.4. The above proof can be slightly altered to classify pivotal categories generated by an object of Frobenius-Perron dimension less than 2, without a unitarity condition. The classification now includes all Galois conjugates of the A_N, $\mathrm{Ad}(A_{2N})$, D_{2N}, E_6, and E_8 planar algebras.

The aim of this thesis will be to generalise the above classification. We will focus on removing the self-dual condition, though many other generalisations are feasible. While a classification for an arbitrary object is still out of reach, Theorem 1.0.1 gives an approach towards classifying a significant subset. This subset will be the unitary categories generated by a *normal* object of dimension less than 2.

Definition 2.0.16. We say an object X is normal if $X \otimes X^* \cong X^* \otimes X$.

Note that the definition of normal requires no conditions on the isomorphism $X \otimes X^* \to X^* \otimes X$.

For completeness, we communicate a proof of Theorem 1.0.1, supplied by Scott Morrison and Noah Snyder.

Proof of Theorem 1.0.1. Let C be a unitary fusion category tensor generated by a normal object X of dimension less than 2. We form the shaded planar algebra (see [31] for a definition):

$$P_{n,+} := \mathrm{Hom}(\mathbf{1} \to (X \otimes X^*)^{\otimes n}) \qquad\qquad P_{n,-} := \mathrm{Hom}(\mathbf{1} \to (X^* \otimes X)^{\otimes n}).$$

By the classification of subfactors of index less than 4, this shaded planar algebra is actually unshaded, and must be one of the ADE planar algebras.

We now claim that the adjoint subcategory of C is tensor generated by $X \otimes X^*$. Let $Y \in \mathrm{Ad}(C)$, then by definition Y is a sub-object of $Z \otimes Z^*$ for some $Z \in C$. As X tensor generates C, we have that Z is a sub-object of $X^{\otimes k} \otimes X^{*\otimes l}$ for some positive integers k and l. Thus we have that Y is a sub-object of $X^{\otimes k} \otimes X^{*\otimes l} \otimes X^{\otimes l} \otimes X^{*\otimes k}$, and thus, using normality of X, a sub-object of $(X \otimes X^*)^{k+l}$.

Putting everything together we get that $\mathrm{Ad}(C)$ is equivalent to $\langle X \otimes X^* \rangle$. As P has to be an ADE planar algebra we have that $\mathrm{Ad}(C)$ is the adjoint subcategory of an ADE fusion category. □

Chapter 3

The Brauer-Picard groups of the adjoint subcategories of the ADE fusion categories

With Theorem 1.0.1 in mind, we want to classify certain cyclic extensions of the adjoint subcategories of the ADE fusion categories. The recipe for such a classification, as laid out in [17] (and described in Chapter 2), suggests that the first step should be to compute the Brauer-Picard groups of these fusion categories. In this Chapter, we will prove the following Theorem.

Theorem 3.0.1. We have the following:

C		$\mathrm{BrPic}(C)$
$\mathrm{Ad}(A_N)$	$N = 3$	$\mathbb{Z}/2\mathbb{Z}$
	$N = 7$	$D_{2\cdot 4}$
	$N \equiv 0 \pmod 2$	$\{e\}$
	$N \equiv 1 \pmod 4$	$\mathbb{Z}/2\mathbb{Z}$
	$N \equiv 3 \pmod 4$ and $N \neq \{3, 7\}$	$(\mathbb{Z}/2\mathbb{Z})^2$
$\mathrm{Ad}(D_{2N})$	$N = 5$	$(S_3)^2$
	$N \neq 5$	$(\mathbb{Z}/2\mathbb{Z})^2$
$\mathrm{Ad}(E_6^{\pm})$		$\mathbb{Z}/2\mathbb{Z}$
$\mathrm{Ad}(E_8^{\pm})$		$\mathbb{Z}/2\mathbb{Z}$.

For the examples we are interested in it turns out to be quite difficult to compute the invertible bimodules explicitly, such as was done in [24]. Thankfully the isomorphism of groups from [17]:

$$\mathrm{BrPic}(C) \cong \mathrm{Aut}^{\mathrm{br}}(Z(C)), \tag{3.0.1}$$

allows an alternate method to compute the Brauer-Picard group of a fusion category. Here $\mathrm{Aut}^{\mathrm{br}}(Z(C))$ is the group of braided auto-equivalences of the Drinfeld centre of C. We find the latter group much easier to compute, and hence spend this Chapter computing Drinfeld centres and braided auto-equivalence groups, rather than a direct computations of invertible bimodules. Although in a few difficult cases we use both descriptions, utilising the relative strengths of both.

Our main tool to compute braided auto-equivalences of categories is via the braided automorphisms of their associated braided planar algebras. It has long been known that there is a one-to-one correspondence between pivotal categories with a distinguished generating object, and planar algebras. This correspondence turns out to be functorial. In particular we have the following:

Proposition 3.0.1. *Let P a be (braided) planar algebra, and C_P be the associated pivotal (braided) tensor category with distinguished generating object X. Let $\mathrm{Aut}^{piv}(C_P; X)$ be the group of pivotal (braided) auto-equivalences of C_P that fix X on the nose (see [27] for a definition of pivotal functors), and $\mathrm{Aut}(P)$ be the group of (braided) planar algebra automorphisms of P. Then there is an isomorphism of groups*

$$\mathrm{Aut}(P) \xrightarrow{\sim} \mathrm{Aut}^{piv}(C_P; X).$$

This proposition is a direct consequence of [27, Theorem 2.4]. While the fact that planar algebra automorphisms only give us pivotal auto-equivalences of the corresponding category may seem restrictive, it turns out that all of the auto-equivalences we care about in this Chapter are pivotal (although not a priori). In particular the gauge auto-equivalences (auto-equivalences whose underlying functor is the identity) of a pivotal category are always pivotal functors when the category has trivial universal grading group, or when every object of the category is self-dual. These two cases include every category we consider in this Chapter. While it seems somewhat reasonable to expect gauge auto-equivalences are always pivotal, regardless of extra assumptions on the category, the author was unable to produce a general proof.

3.1 The Drinfeld Centres of the ADE fusion categories

As stated earlier, the main aim of this Chapter is to compute the Brauer-Picard groups of the adjoint subcategories of the ADE fusion categories. We do this via computing the braided auto-equivalence groups of the Drinfeld centre. This Section is devoted to computing these centres. With Proposition 3.0.1 in mind is natural to want planar algebra presentations of the centres.

The centres of the full ADE fusion categories are computed in [6, 15, 9].

$$\begin{aligned} Z(A_N) &\simeq A_N \boxtimes A_N^{\text{bop}}, \\ Z(D_{2N}^{+}) &\simeq A_{4N-3} \boxtimes \text{Ad}(D_{2N})^{\text{bop}}, \\ Z(E_6^{+}) &\simeq A_{11} \boxtimes A_3, \\ Z(E_8^{+}) &\simeq A_{29} \boxtimes \text{Ad}(A_4)^{\text{bop}}. \end{aligned}$$

Note that we have given planar algebra presentations for the factors of these products in Chapter 2.

The centres of the adjoint subcategories have yet to be described in the literature. When the adjoint subcategory admits a modular braiding the centres are trivial to compute.

Lemma 3.1.1.

$$\begin{aligned} Z(\text{Ad}(A_{2N})) &\simeq \text{Ad}(A_{2N}) \boxtimes \text{Ad}(A_{2N})^{\text{bop}} \\ Z(\text{Ad}(D_{2N})) &\simeq \text{Ad}(D_{2N}) \boxtimes \text{Ad}(D_{2N})^{\text{bop}} \end{aligned}$$

Proof. These fusion categories admit modular braidings, thus by a theorem of Muger [43] the centres are as described. □

To compute the centres of the rest of the adjoint subcategories we appeal to a theorem of Gelaki, Naidu and Nikshych (stated below) which allows us to compute the centre of $\text{Ad}(C)$ given the centre of C. As we know the centres of all the ADE fusion categories we are able to get explicit descriptions of the centres of the corresponding adjoint subcategories. We explicitly describe the details this theorem as we will be using it extensively for the rest of this Section.

Construction 3.1.2. [21, Corollary 3.7] Let C be a G-graded fusion category, with trivially graded piece C_0. Consider all objects in the centre of C that restrict to direct sums of the tensor identity in C. These objects form a subcategory equivalent to $\mathrm{Rep}(G)$. If we take the centraliser (see [43] for details) of the $\mathsf{Rep}(G)$ subcategory of $Z(C)$, and de-equivariantize by the distinguished copy of $\mathrm{Rep}(G)$, then the resulting category is braided equivalent to $Z(C_0)$.

Straight away we can use this construction to get an explicit description of the centre of $\mathrm{Ad}(E_8^+)$.

Lemma 3.1.3. $Z(\mathrm{Ad}(E_8^+)) \simeq \mathrm{Ad}(D_{16}) \boxtimes \mathrm{Ad}(A_4)^{\mathrm{bop}}$

Proof. As there are only two simple objects of dimension 1 in $A_{29} \boxtimes \mathrm{Ad}(A_4)^{\mathrm{bop}}$ it follows that the $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$ subcategory we are interested in must be the $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$ subcategory of A_{29}. We compute the centralizer of this subcategory to be $\mathrm{Ad}(A_{29}) \boxtimes \mathrm{Ad}(A_4)^{\mathrm{bop}}$. The result now follows as de-equivariantizing commutes with the taking of the adjoint subcategory, and $A_{29}// \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z}) = D_{16}$. □

This Lemma also appeared with slightly different language in [5].

Remark 3.1.4. While the centre of $\mathrm{Ad}(E_8^-)$ is also monoidally equivalent to $\mathrm{Ad}(D_{16}) \boxtimes \mathrm{Ad}(A_4)^{\mathrm{bop}}$, in this case the equivalence is not braided (for our choice of braidings on $\mathrm{Ad}(D_{16})$ and $\mathrm{Ad}(A_4)$). Infact it is braided equivalent to $\mathrm{Ad}(D_{10}^{\mathrm{bop}}) \boxtimes \mathrm{Ad}(A_4)$.

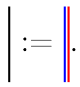
Planar Algebras for the Drinfeld Centres of $\mathrm{Ad}(A_{2N+1})$ and $\mathrm{Ad}(E_6^+)$

While we can apply the same techniques as in computing $Z(\mathrm{Ad}(E_8^+))$ to computing $Z(\mathrm{Ad}(A_{2N+1}))$ and $Z(\mathrm{Ad}(E_6^+))$, the resulting categories don't naively have a planar algebra presentation. To achieve such a presentation we work through Construction 3.1.2 on the level of the planar algebra.

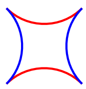
We start by looking at the $\mathrm{Ad}(A_{2N+1})$ family. The $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$ subcategory in $A_{2N+1} \boxtimes A_{2N+1}^{\mathrm{bop}}$ that we are interested in is generated by the simple object $f^{(2N)} \boxtimes f^{(2N)}$. The centralizer of this subcategory is the subcategory of $A_{2N+1} \boxtimes A_{2N+1}^{\mathrm{bop}}$ generated by the simple object $f^{(1)} \boxtimes f^{(1)}$. Note that $\langle f^{(1)} \boxtimes f^{(1)} \rangle$ is the tensor product planar algebra of A_{2N+1} with A_{2N+1}^{bop} (see [31, Section 2.3] for details on the tensor product of planar algebras).

The planar algebra for $\langle f^{(1)} \boxtimes f^{(1)} \rangle$ can easily be described using basis vectors for the vector spaces. Let $\{t_i\}$ be the standard Temperley-Lieb basis of planar diagrams for P_n of the planar algebra associated to A_{2N+1}. Then a basis for P_n for the planar algebra of $\langle f^{(1)} \boxtimes f^{(1)} \rangle$ are the vectors $\{t_i \boxtimes t_j\}$. Diagrammatically we can think of these basis vectors as a superposition of the two original basis vectors. Unfortunately this simple basis description of the planar algebra is not useful for computing automorphisms nor for taking de-equivariantizations. We want to find a generators and relations description (as in [2, 40]) as it is much better suited for these tasks.

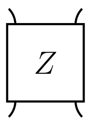
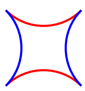
The single strand in the $\langle f^{(1)} \boxtimes f^{(1)} \rangle$ planar algebra is going to be the superposition of the red and blue strands, that is:



Unfortunately just the strand does not generate the entire planar algebra, for example we can not construct the diagram

 $\in P_4$

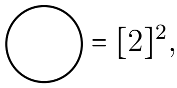
using just the single strand. However it is proven in [38, Theorem 3.1] that the element

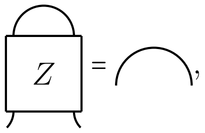
 $:= [2]_q^{-1}$  $\in P_4$

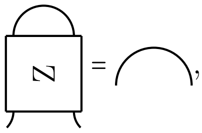
does generate everything.

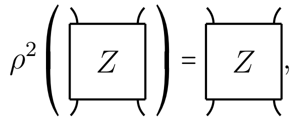
A generators and relations presentation of the braided planar algebra for $\langle f^{(1)} \boxtimes f^{(1)} \rangle$ is given by

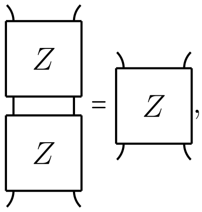
(1) $q = e^{\frac{\pi i}{2N+2}}$,

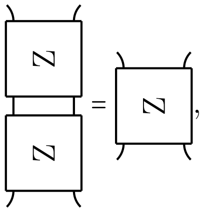
(2) 

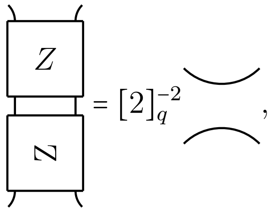
(3) 

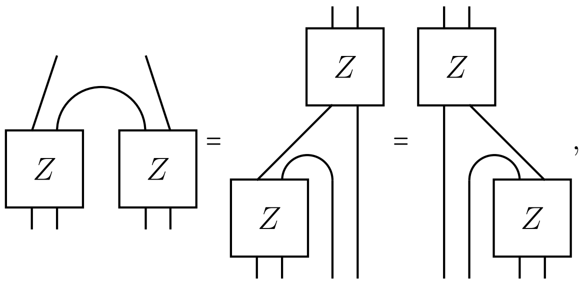
(4) 

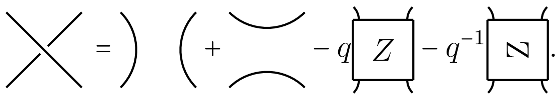
(5) 

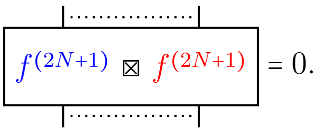
(6) 

(7) 

(8) 

(9) 

(11) 

(12) 

All of these relations can easily be checked to hold in $\langle f^{(1)} \boxtimes f^{(1)} \rangle$ by using the definition of Z. We have certainly overdone the number the relations required for this planar algebra. For example relations (3) + (5) + (9) imply relation (6). We have included the additional relations as it makes evaluation easier and there is little overhead in showing that the extra relations hold for Z.

To prove that we have given sufficient relations we need to show that we can evaluate any closed diagram. We can think of a closed diagram in the $\langle f^{(1)} \boxtimes f^{(1)} \rangle$ planar algebra as a 4-valent graph, with the vertices being Z's or $\rho(Z)$'s. Any closed 4-valent graph must contain either a triangle, bigon or a loop. We can remove a loop with relations 3 and 4. We can pop bigons with relations 6, 7 and 8. Thus to complete our evaluation argument all we need to do is show that triangles can be popped as well.

Lemma 3.1.5. Suppose D is a diagram containing three Z's as a triangle, then using the above relations D can be reduced to a diagram with at most two Z's.

Proof. In D two of the Z's must form one of the diagrams appearing in relation 9. Locally apply the relation to D to obtain a diagram with two Z's forming a bigon. Pop this bigon using one of relations 6, 7, or 8 to obtain a diagram with one or two Z's. □

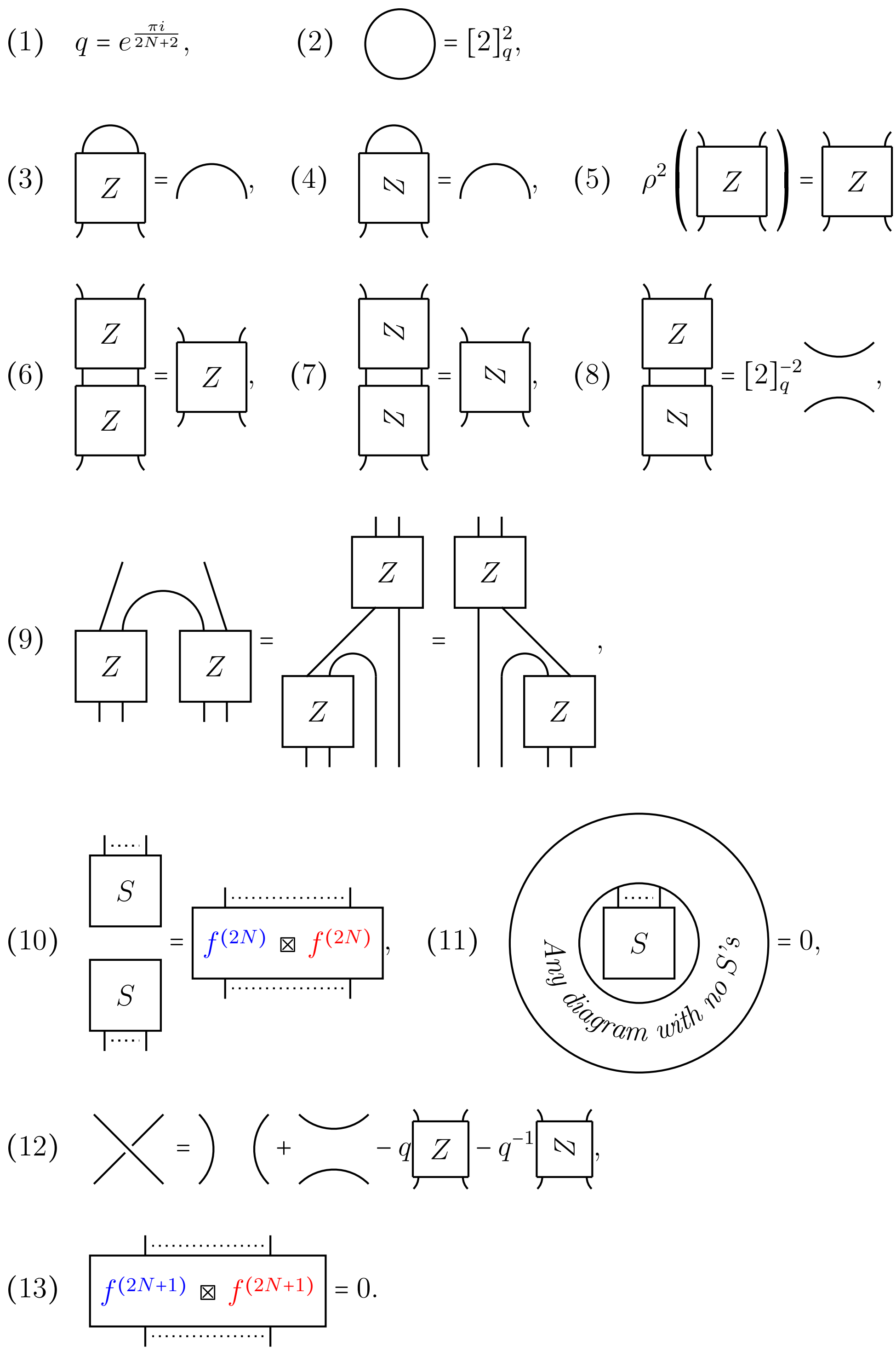
By repeatedly popping triangles, bigons, and closed loops eventually one ends up at a scalar multiple of the empty diagram.

To de-equivariantize the $\langle f^{(1)} \boxtimes f^{(1)} \rangle$ planar algebra by the subcategory $\langle f^{(2N)} \boxtimes f^{(2N)} \rangle$ we need to add an isomorphism from the tensor identity to $f^{(2N)} \boxtimes f^{(2N)}$. In planar algebra language this corresponds to adding a generator $S \in P_{2N}$ such that $SS^{-1} = f^{(2N)} \boxtimes f^{(2N)} \in P_{4N}$ and $S^{-1}S = 1 \in P_0$. Note that by taking the trace of the first condition we arrive at the second.

See [40] for an example of this in which they de-equivariantize the A_{4N-3} planar algebra to obtain a planar algebra for D_{2N}. By doing the same procedure we arrive at the following presentation of the planar algebra for the centre of $\mathrm{Ad}(A_{2N+1})$.

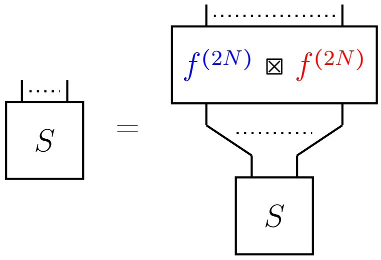
Proposition 3.1.1. *The braided planar algebra for the Drinfeld centre of* $\mathrm{Ad}(A_{2N+1})$

has 2 generators $Z \in P_4$ and $S \in P_{2N}$ satisfying the following relations:



Again we have almost certainly overdone the relations to obtain a nice evaluation algorithm. However now it is not so obvious why relation (11) should exist in this planar algebra.

Consider a diagram D that only contains a single S. By attaching an S to the bottom of relation (10) we get the relation:



Therefore in D we can fit a $f^{(2N)} \boxtimes f^{(2N)}$ between the S and the rest of the diagram. Consider the blue component of the diagram. There must be a blue cap on $f^{(2N)}$ which kills the diagram.

The reader might argue that relation (11) isn't really a satisfying relation, as it really involves an infinite number of diagrams. The relation is also non-local as it is defined on entire diagrams. It appears that in the $Z(\mathrm{Ad}(A_{2N+1}))$ planar algebra one can deduce relation (11) from relation (10). However this argument is fairly lengthy. As the above presentation is sufficient for our purposes in this Chapter we omit this argument and work with the less satisfying relations. As we will see soon, the planar algebra for $Z(\mathrm{Ad}(E_6^+))$ has a similar unsatisfying relation. For this case we are unsure if the relation can be deduced from the others, though we suspect that it can.

Proof of Proposition 3.1.1. To show that we have given enough relations we have to show that any closed diagram can be evaluated to a scalar. If we have a diagram with multiple S's, then using the fact that the planar algebra is braided along with relation (10) we can reduce to a diagram with 0 or 1 S's. See [40] for an in depth description on how this algorithm works. We have already shown that any diagram with 0 S's can be evaluated. Relation (11) exactly says that any diagram with 1 S is evaluated to 0 as in such a diagram there must be a cup between two adjacent strands of S.

We now claim that this planar algebra is exactly the planar algebra for the de-equivariantization of $\langle f^{(1)} \boxtimes f^{(1)} \rangle$ by the copy of $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z}) \simeq \langle f^{(2N)} \boxtimes f^{(2N)} \rangle$. To see this notice that the above planar algebra has an order two automorphism $S \mapsto -S$. Taking the fixed point planar algebra under this action of $\mathbb{Z}/2\mathbb{Z}$ gives the sub-planar algebra consisting of elements with an even number of S's. However the braiding along with relation (10) implies that such a diagram is equal to a diagram with no S's. Hence the fixed point planar algebra recovers the planar algebra for $\langle f^{(1)} \boxtimes f^{(1)} \rangle$.

Theorem 4.44 of [12] states that braided fusion categories containing $\mathrm{Rep}(G)$, and G-crossed braided fusion categories, are in bijection via equivariantization, and de-equivariantization. As the braided tensor category associated to the above planar algebra equivariantizes to give $\langle f^{(1)} \boxtimes f^{(1)} \rangle$, it follows that the braided tensor category associated to the above planar algebra is a de-equivariantization of $\langle f^{(1)} \boxtimes f^{(1)} \rangle$ by some copy of $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$. We have to show that the copy of $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$ is precisely $\langle f^{(2N)} \boxtimes f^{(2N)} \rangle$.

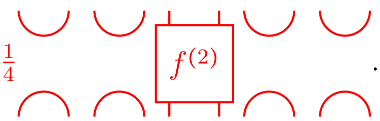
When N is odd there is a unique copy of $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$ in $\langle f^{(1)} \boxtimes f^{(1)} \rangle$ and so the claim is trivial. When N is even there are 3 distinct copies, generated by the objects $f^{(0)} \boxtimes f^{(2N)}$, $f^{(2N)} \boxtimes f^{(0)}$, and $f^{(2N)} \boxtimes f^{(2N)}$. The de-equivariantizations corresponding to these subcategories are $\langle f^{(1)} \boxtimes f^{(1)} \rangle \subset A_{2N+1} \boxtimes D_{N+2}^{\mathrm{op}}$, $\langle f^{(1)} \boxtimes f^{(1)} \rangle \subset D_{N+2} \boxtimes A_{2N+1}^{\mathrm{op}}$, and $Z(\mathrm{Ad}(A_{2N+1}))$.

The former two categories don't admit braidings, and hence can't come from the above planar algebra. Thus the corresponding fusion category for the above planar algebra is $Z(\mathrm{Ad}(A_{2N+1}))$.

□

Note that when $N = 1$ we have a generators and relations presentation of the planar algebra associated to $Z(\mathsf{Rep}(\mathbb{Z}/2\mathbb{Z}))$, the toric code.

The computation for the centre of $\mathrm{Ad}(E_6^+)$ is almost identical, except we now start with the $A_{11} \boxtimes A_3^{\mathrm{bop}}$ planar algebra. The only difference in the computation is that the object $f^{(10)} \boxtimes f^{(2)}$ doesn't naively make sense in planar algebra language. To deal with this one has to choose an isomorphic copy of $f^{(2)}$ living in P_{20} boxspace of the A_3^{bop} planar algebra. For example you can choose the projection:

$\frac{1}{4}$ 

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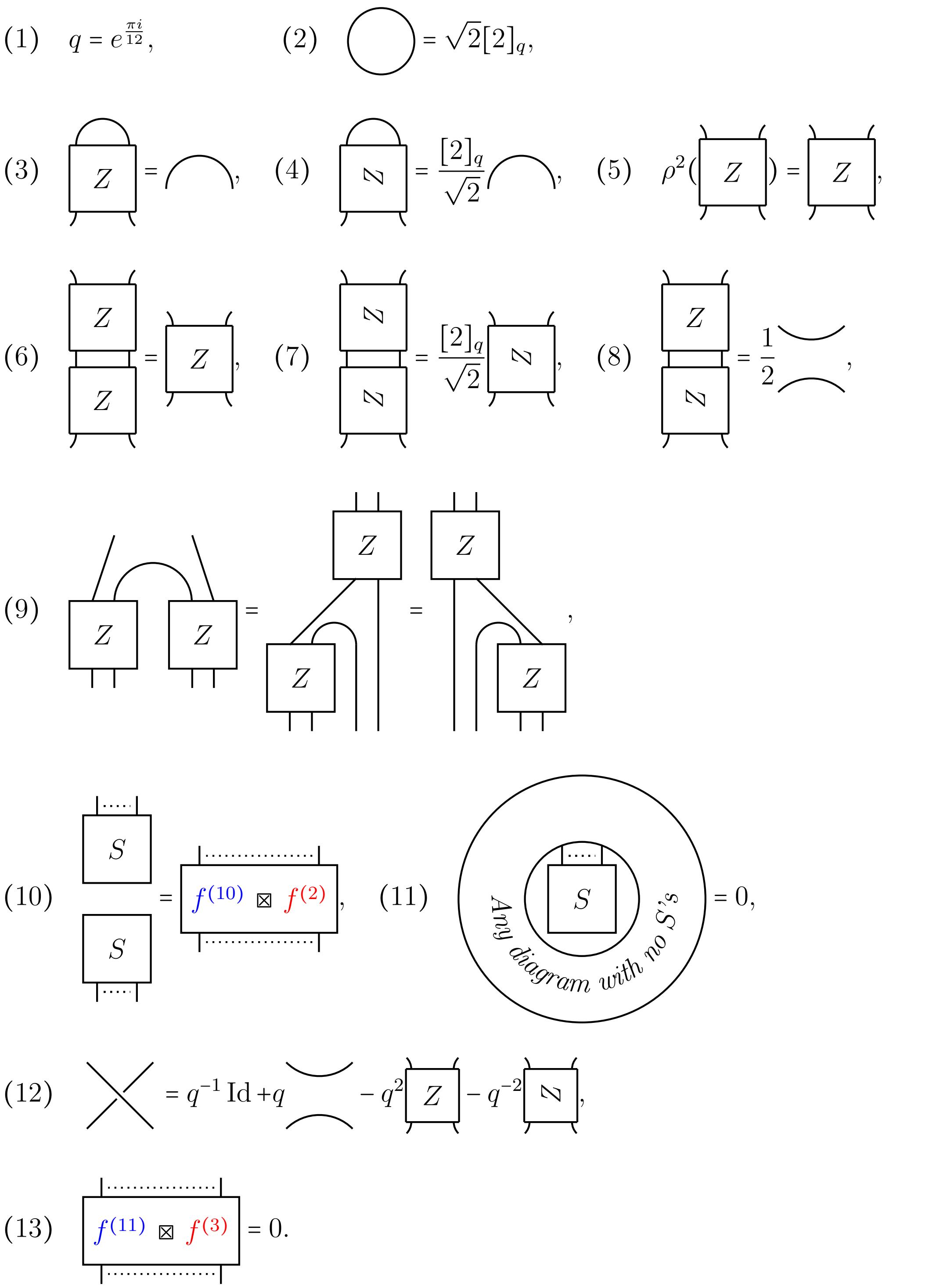
The resulting planar algebra for the centre is as follows:

Proposition 3.1.2. *The braided planar algebra for the Drinfeld centre of* $\mathrm{Ad}(E_6^+)$

has 2 generators $Z \in P_4$ and $S \in P_{10}$ satisfying the following relations:



Proof. As mentioned this proof is almost identical to the $Z(\mathrm{Ad}(A_{2N+1}))$ case. In fact there is a unique copy of $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$ in $A_{11} \boxtimes A_3^{\mathrm{bop}}$ which causes this case to be simpler. $\square$

Remark 3.1.6. Note that we haven't explicitly given descriptions of $f^{(2N)} \boxtimes f^{(2N)}$ and $f^{(10)} \boxtimes f^{(2)}$ in terms of the generator Z. This is because these descriptions are particularly nasty to write down. We just appeal to the existence of such a description from [36, Theorem 4]. If the reader requires a better description one can be obtained by applying the Jones-Wenzl recursion formula to both idempotents and rewriting the resulting recursive formula in terms of Z's and $\rho(Z)$'s.

3.2 Planar algebra automorphisms and auto-equivalences of tensor categories

Planar Algebra Automorphisms

As mentioned earlier in this Chapter we wish to compute the auto-equivalence group of a fusion category by studying the associated planar algebra. In [27] the authors show an equivalence between the categories of (braided) planar algebras and the category of (braided) pivotal categories with specified generating object. As described in that paper the latter category is really a 2-category, with 1-morphisms being pivotal functors that fix the specified generating object up to isomorphism. The 2-morphisms are natural transformations such that the component on the generating object is the identity of the generating object. It is shown that there is at most one 2-morphism between any pair of 1-morphisms, and further that it must be an isomorphism. Thus there is no loss of generality in truncating to a 1-category. See [27, Definition 3.4] for details. We specialise their result as Proposition 3.0.1.

The (braided) fusion categories that we wish to compute the auto-equivalence groups for satisfy the necessary conditions to apply the above equivalence of categories (admit pivotal structures and generated by a specified object). However the restriction that the component of a natural transformation is the identity turns out to be too strong. Thus planar algebra homomorphisms which are non-equal can be mapped to isomorphic functors of the associated pivotal tensor categories. To fix this we have to work out explicitly which planar algebra automorphisms are mapped to isomorphic auto-equivalences.

The aim of this Chapter is to compute the Brauer-Picard groups of the ADE fusion categories via braided auto-equivalences of their centres. The above Theorem suggests that we should begin by studying the braided automorphisms of the planar algebras associated to the centres. To remind the reader these are

A_N, $\mathrm{Ad}(A_{2N})$, $\mathrm{Ad}(D_{2N})$, $Z(\mathrm{Ad}(A_{2N+1}))$, and $Z(\mathrm{Ad}(E_6)^+)$. Note that some of the centres appear as Deligne products of these categories e.g. $Z(\mathrm{Ad}(D_{2N})) \simeq \mathrm{Ad}(D_{2N}) \boxtimes \mathrm{Ad}(D_{2N})^{\mathrm{bop}}$. We deal with such products in Section 3.3.

Roughly speaking a planar algebra automorphism is a collection of vector space automorphisms, one for each box space, that commute with the action of planar tangles. More details on planar algebra automorphisms can be found in [31]. As we have described the planar algebras we are interested in terms of generators and relations it is enough to determine how the automorphisms act on the generators, as any element of the planar algebra will be a sum of diagrams of generators connected by planar tangle. This makes the A_N case particularly easy as this planar algebra has no generators.

Lemma 3.2.1. The planar algebras for A_N have no non-trivial automorphisms.

Proof. Any diagram in the A_N planar algebra is entirely planar tangle. As automorphisms of planar algebras have to commute with planar tangles, only the identity automorphism can exist. □

Unfortunately the other cases are not as easy.

Lemma 3.2.2. There are two braided planar algebra automorphisms of $\mathrm{Ad}(A_N)$. When lifted to the braided fusion category $\mathrm{Ad}(A_N)$ the corresponding auto-equivalences are naturally isomorphic.

Proof. Let ϕ be a planar algebra automorphism of $\mathrm{Ad}(A_N)$. As the $\mathrm{Ad}(A_N)$ planar algebra has the single generator $T \in P_3$ it is enough to consider how ϕ behaves on T. As P_3 is one dimensional it follows that $\phi(T) = \alpha T$ for some $\alpha \in \mathbb{C}$. As ϕ must fix the single strand we can apply ϕ to relation (5) to show that $\alpha \in \{1, -1\}$. It can be verified that $\phi(T) = -T$ is consistent with the other relations and hence determines a valid planar algebra automorphism.

We now claim that the auto-equivalence of the $\mathrm{Ad}(A_N)$ fusion category generated by this planar algebra automorphism is naturally isomorphic to the identity. Let (n_1, p_1) and (n_2, p_2) be objects of the $\mathrm{Ad}(A_N)$ fusion category, where $n_i \in \mathbb{N}$ and p_i are projections in P_{n_i}. We define the components of our natural isomorphism to be $\tau_{(n,p)} := (-1)^n$. We need to verify that the following diagram commutes for any morphism $f : (n_1, p_1) \to (n_2, p_2)$:

$$\begin{tikzcd}
(n_1, p_1) \arrow[r, "\phi(f)"] \arrow[d, "(-1)^{n_1}"] & (n_2, p_2) \arrow[d, "(-1)^{n_2}"] \\
(n_1, p_1) \arrow[r, "f"] & (n_2, p_2)
\end{tikzcd}$$

Recall that $f \in P_{n_1+n_2}$. First suppose that $n_1 + n_2$ is even, then the number of T generators in f must be even, thus $\phi(f) = f$. However in this case we also have that $(-1)^{n_1} = (-1)^{n_2}$ and so the above diagram commutes. If $n_1 + n_2$ is odd, then the number of T generators in f must be odd, thus $\phi(f) = -f$. However in this case we also have that $(-1)^{n_1} = -(-1)^{n_2}$ and so the above diagram commutes. $\square$

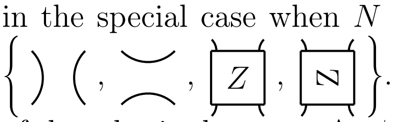
Lemma 3.2.3. There are four braided planar algebra automorphisms of $\mathrm{Ad}(D_{2N})$. When lifted to the braided fusion category $\mathrm{Ad}(D_{2N})$ two of the auto-equivalences are naturally isomorphic to the other two. Furthermore the non-trivial automorphism lifts to a non-gauge braided auto-equivalence of the category $\mathrm{Ad}(D_{2N})$.

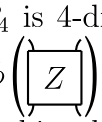
Proof. Let ϕ be a planar algebra automorphism of $\mathrm{Ad}(D_{2N})$. Recall the planar algebra $\mathrm{Ad}(D_{2N})$ has two generators $T \in P_3$ and $S \in P_{2N-2}$. As P_3 is one dimensional it follows that $\phi(T) = \alpha T$ for some $\alpha \in \mathbb{C}$. Applying relation (7) shows us that $\alpha \in \{1, -1\}$.

The vector space P_{2N-2} can be written $TL_{2N-2} \oplus \mathbb{C}S$, thus $\phi(S) = f + \beta S$ where $f \in TL_{2N-2}$ and $\beta \in \mathbb{C}$. We can use relation (10) to show that $\phi(S)$ must be uncappable, and hence f must be 0 as there are no non-trivial uncappable Temperley-Lieb elements. Further relation (10) now shows that $\beta \in \{1, -1\}$. It can be verified that any combination of choices of α and β gives a valid planar algebra automorphism. This gives us four automorphisms.

However as monoidal auto-equivalences, the automorphisms which send $T \mapsto -T$ are naturally isomorphic to the corresponding automorphisms which leaves T fixed. The proof of this is identical to the argument used in the previous proof. Thus we get that as a fusion category $\mathrm{Ad}(D_{2N})$ has a single non-trivial auto-equivalence. We can see that this non-trivial auto-equivalence is not a gauge auto-equivalence as it exchanges the simple objects P and Q. $\square$

Lemma 3.2.4. There are two braided automorphisms of the $Z(\mathrm{Ad}(A_{2N+1}))$ planar algebra. The non-trivial automorphism lifts to a non-gauge braided auto-equivalence of $Z(\mathrm{Ad}(A_{2N+1}))$.

Proof. Let ϕ be a braided planar algebra automorphism of $Z(\mathrm{Ad}(A_{2N+1}))$. As a planar algebra $Z(\mathrm{Ad}(A_{2N+1}))$ has two generators $Z \in P_4$ and $S \in P_{2N}$. Except in the special case when $N = 2$ the box space P_4 is 4-dimensional, spanned by 

. Thus we can write ϕ 

as a linear combination of these basis elements. As ϕ is a braided automorphism by definition it preserves the braiding, that is

$$)\,(+ \asymp + q\,\boxed{Z} + q^{-1}\,\boxed{\rotatebox{90}{Z}} =)\,(+ \asymp + q\,\phi\left(\boxed{Z}\right) + q^{-1}\,\phi\left(\boxed{\rotatebox{90}{Z}}\right).$$

Solving this equation gives the unique solution $\phi\left(\boxed{Z}\right) = \boxed{Z}$.

For the $N = 2$ case we repeat the same proof but with P_4 also having S as a basis vector.

Now we have to determine where ϕ sends the generator $S \in P_{2N}$. Recall that relation (11) shows that any diagram with a single S is zero. We claim that S is the only vector (up to scalar) in P_{2N} with this property. Let $v \in P_N$ be such a vector, as $P_{2N} \cong TL_{2N} \boxtimes TL_{2N} \oplus \mathbb{C}S$ we can write $v = f_1 \boxtimes f_2 + \alpha S$ where f_1 and f_2 are elements of TL_{2N}. As αS and v are uncappable, it follows that $f_1 \boxtimes f_2$ must also be. In particular this implies that both f_1 and f_2 are uncappable Temperley-Lieb diagrams, and so must be 0. Thus $v = \alpha S$.

Relation (10) now implies that $\phi(S) = \pm S$ as we know ϕ fixes Z and hence the right hand side of the relation. It can be verified that $S \mapsto -S$ is consistent with the other relations. The automorphism $\phi(S) = -S$ lifts to a non-gauge braided auto-equivalence of the associated category as it exchanges the simple objects $\frac{f^{(N)}+S}{2}$ and $\frac{f^{(N)}-S}{2}$. □

Lemma 3.2.5. There are two braided automorphisms of the $Z(\mathrm{Ad}(E_6^+))$ planar algebra. The non-trivial automorphism lifts to a non-gauge braided auto-equivalence of $Z(\mathrm{Ad}({E_6}^+))$.

Proof. Almost identical to the proof of Lemma 3.2.4. □

Braided auto-equivalences

The aim of this subsection is to leverage our knowledge of planar algebra automorphisms to compute the braided auto-equivalence group of the associated braided category. As mentioned in the introduction, for our examples, planar algebra automorphisms contain the gauge auto-equivalences of the associated braided category. The results of the previous section show that there are in fact no non-trivial gauge auto-equivalences for any of the categories we are interested in! This means that the group of braided auto-equivalences is a subgroup of the automorphism group of the fusion ring.

Our proofs to compute braided auto-equivalence group of C is as follows. First we compute the fusion ring automorphisms of C. Then we analyse the t-values of the simple objects of C to rule out fusion ring automorphisms that can't lift to braided auto-equivalences of C. Finally we construct braided auto-equivalences of C to realise the remaining fusion ring automorphisms.

Lemma 3.2.6. We have $\mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(A_{2N})) = \{e\}$.

Proof. The fusion ring of $\mathrm{Ad}(A_{2N})$ has no non-trivial automorphisms. Thus in light of Theorem 3.0.1 and Lemma 3.2.2 there are no monoidal auto-equivalences of the category $\mathrm{Ad}(A_{2N})$, and in particular no braided auto-equivalences. □

Lemma 3.2.7. We have

$$\mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(D_{2N})) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{when } N \neq 5 \\ S_3 & \text{when } N = 5. \end{cases}$$

Proof. We break this proof up into three parts: $N = 2$, $N = 5$ and all other N.

Case $N = 2$**:**

The category $\mathrm{Ad}(D_4)$ has fusion ring isomorphic to that of $\mathrm{Vec}(\mathbb{Z}/3\mathbb{Z})$. It is straightforward to see that the only possible fusion ring automorphism exchanges the two non-trivial objects. Lemma 3.2.3 tells us two pieces of information. First is that there are no gauge auto-equivalences of $\mathrm{Ad}(D_4)$, and thus there are at most two braided auto-equivalences of $\mathrm{Ad}(D_4)$. Second is that there exists a non-trivial braided auto-equivalence of $\mathrm{Ad}(D_4)$, which realises the upper bound on the braided auto-equivalence group.

Case $N = 5$**:** Studying the $\mathrm{Ad}(D_{10})$ fusion ring we see that there are 6 possible automorphisms, corresponding to any permutation of the objects $f^{(2)}$, P, and Q. From Lemma 3.2.3 we see that there are no non-trivial gauge auto-equivalences, thus $\mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(D_{2N}))$ is a subgroup of S_3.

To show that $\mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(D_{10})) = S_3$ we notice from Lemma 3.2.3 that $\mathrm{Ad}(D_{10})$ has an order 2 braided auto-equivalence that exchanges the objects P and Q, and from [41, Theorem 4.3] $\mathrm{Ad}(D_{10})$ has an order 3 braided auto-equivalence. Therefore Lagrange's Theorem implies that the order of $\mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(D_{10}))$ is at least 6 and the result follows.

Case $N > 2, N \neq 5$**:** For these cases, the only simple objects with the same dimension are P and Q. Thus there can be at most two fusion ring automorphisms of the $\mathrm{Ad}(D_{2N})$ fusion ring. The result now follows by the same argument as in the $N = 2$ case. □

Lemma 3.2.8. We have

$$\mathrm{Aut}^{\mathrm{br}}(A_N) = \begin{cases} \{e\} & N \equiv \{1,2,4\} \pmod 4 \\ \mathbb{Z}/2\mathbb{Z} & N \equiv 3 \pmod 4. \end{cases}$$

Proof. When N is odd, the A_N fusion ring has the non-trivial automorphism:

$$f^{(n)} \mapsto \begin{cases} f^{(n)} & n \text{ is even} \\ f^{(N-n-1)} & n \text{ is odd.} \end{cases}$$

Case $N \equiv \{0,2\} \pmod 4$: When N is even there are no non-trivial automorphisms of the A_N fusion ring. Thus the result follows as there are no non-trivial gauge auto-equivalences of the category A_N.

Case $N \equiv 1 \pmod 4$**:**

In this case there is a fusion ring automorphism exchanging the generating object $f^{(1)}$ and $f^{(N-2)}$, however these two objects have different t-values so there is no braided auto-equivalence realising the fusion ring automorphism. Thus the result follows as there are no non-trivial gauge auto-equivalences of the category A_N.

Case $N \equiv 3 \pmod 4$**:**

We first give a bound on the size of the auto-equivalence group of A_N. On the level of fusion rings there are two automorphisms, with the non-trivial one exchanging $f^{(1)}$ and $f^{(N-2)}$. As there are no non-trivial gauge auto-equivalences of the category A_N, we have that $\mathrm{Aut}^{\mathrm{br}}(A_N) \subseteq \mathbb{Z}/2\mathbb{Z}$.

To show existence of the non-trivial braided auto-equivalence it suffices to show that the category generated by $f^{(N-2)}$ is equivalent to A_N as a braided tensor category. It is known that the T-matrix is a complete invariant of braided fusion categories with A_N fusion rules [18]. The braided fusion category generated by $f^{(N-2)}$ has the same T-matrix as A_N, hence they are braided equivalent. □

We can slightly modify the above argument to also compute the tensor auto-equivalences of A_N.

Lemma 3.2.9. We have

$$\mathrm{Aut}_{\otimes}(A_N) = \begin{cases} \{e\} & N \equiv 0 \pmod 2 \\ \mathbb{Z}/2\mathbb{Z} & N \equiv 1 \pmod 2. \end{cases}$$

Proof. When N is even there are no non-trivial automorphisms of the A_N fusion ring. Thus as there are no automorphisms of the A_N planar algebra, we have $\mathrm{Aut}_\otimes(A_N) = \{e\}$.

When N is odd the same argument as we used in Lemma 3.2.8 shows that $\mathrm{Aut}(A_N) \subseteq \mathbb{Z}/2\mathbb{Z}$. To realise the non-trivial tensor auto-equivalence we need to show that the tensor category generated by $f^{(N-2)}$ is equivalent to A_N. Recall that the categorical dimension of the generating object of A_N is a complete invariant of these tensor categories [18]. The result then follows as $f^{(1)}$ and $f^{(N-2)}$ have the same categorical dimension. □

While the $\mathrm{Ad}(A_{2N+1})$ categories do not appear as factors of any of the centres we are studying in this Chapter, we can show the existence of an exceptional monoidal auto-equivalence of $\mathrm{Ad}(A_7)$. This auto-equivalence will be useful when trying later when trying to construct invertible bimodules over $\mathrm{Ad}(A_7)$.

Lemma 3.2.10. We have $\mathrm{Aut}_\otimes(\mathrm{Ad}(A_7)) = \mathbb{Z}/2\mathbb{Z}$.

Proof. From the previous subsection we know that there are no gauge auto-equivalences of $\mathrm{Ad}(A_7)$. A quick analysis of the fusion ring of $\mathrm{Ad}(A_7)$ reveals a single non-trivial automorphism, exchanging $f^{(2)}$ and $f^{(4)}$.

Consider the planar algebras

$$\mathrm{PA}(\mathrm{Ad}(A_7); f^{(2)}) \qquad \text{and} \qquad \mathrm{PA}(\mathrm{Ad}(A_7); f^{(4)}).$$

Both of these planar algebras contain sub-planar algebras generated by the trivalent vertex $f^{(2)} \otimes f^{(2)} \to f^{(2)}$ and $f^{(4)} \otimes f^{(4)} \to f^{(4)}$ respectively. By considering the fusion rules for $\mathrm{Ad}(A_7)$ we can see that both these sub-planar algebras have box space dimensions $(1, 0, 1, 1, 3, ...)$, thus by the main Theorem of [42] must be $\mathrm{SO}(3)_q$ for q either $e^{\frac{i\pi}{4}}$ or $e^{\frac{3i\pi}{4}}$. As the categorical dimension of both $f^{(2)}$ and $f^{(4)}$ in $\mathrm{Ad}(A_7)$ is $1 + \sqrt{2}$, we must have that both sub-planar algebras are $\mathrm{SO}(3)_{e^{\frac{i\pi}{4}}}$.

By again considering fusion rules for $\mathrm{Ad}(A_7)$ we can see that

$$\mathrm{PA}(\mathrm{Ad}(A_7); f^{(4)}) = \mathrm{SO}(3)_{e^{\frac{i\pi}{4}}} = \mathrm{PA}(\mathrm{Ad}(A_7); f^{(2)}).$$

Thus there is a planar algebra isomorphism

$$\mathrm{PA}(\mathrm{Ad}(A_7); f^{(2)}) \to \mathrm{PA}(\mathrm{Ad}(A_7); f^{(4)}),$$

which by Proposition 3.0.1 gives us a monoidal equivalence between based categories $(\mathrm{Ad}(A_7), f^{(2)})$ and $(\mathrm{Ad}(A_7), f^{(4)})$. Forgetting the basing realises the non-trivial monoidal auto-equivalence of $\mathrm{Ad}(A_7)$. □

Lemma 3.2.11. We have $\mathrm{Aut}^{\mathrm{br}}(Z(\mathrm{Ad}(E_6^+))) = \mathbb{Z}/2\mathbb{Z}$.

Proof. Recall that $Z(\mathrm{Ad}(E_6))$ is a de-equivariantization of a sub-category of $A_{11} \boxtimes A_3^{\mathrm{bop}}$. Thus we can pick representatives of the simple objects of $Z(\mathrm{Ad}(E_6))$ as in Table 3.1. We include the dimensions and twists of these simple objects (from formulas in [15, 48]).

X	$\dim(X)$	t_X
$f^{(0)} \boxtimes f^{(0)}$	1	1
$f^{(2)} \boxtimes f^{(0)}$	$1+\sqrt{3}$	$e^{\frac{2\pi i}{6}}$
$f^{(4)} \boxtimes f^{(0)}$	$2+\sqrt{3}$	-1
$f^{(6)} \boxtimes f^{(0)}$	$2+\sqrt{3}$	1
$f^{(8)} \boxtimes f^{(0)}$	$1+\sqrt{3}$	$e^{\frac{8\pi i}{6}}$
$f^{(10)} \boxtimes f^{(0)}$	1	-1
$f^{(1)} \boxtimes f^{(1)}$	$1+\sqrt{3}$	$-i$
$f^{(3)} \boxtimes f^{(1)}$	$3+\sqrt{3}$	1
$\frac{1}{2}(f^{(5)} \boxtimes f^{(1)} + S)$	$1+\sqrt{3}$	$e^{\frac{5\pi i}{6}}$
$\frac{1}{2}(f^{(5)} \boxtimes f^{(1)} - S)$	$1+\sqrt{3}$	$e^{\frac{5\pi i}{6}}$

Table 3.1: Dimensions and t-values for the simple objects of $Z(\mathrm{Ad}(E_6^+))$

By considering dimensions and twists, we can see that there is only one possible non-trivial fusion ring automorphism of $Z(\mathrm{Ad}(E_6^+))$, that exchanges the objects $\frac{1}{2}(f^{(5)} \boxtimes f^{(1)} + S)$ and $\frac{1}{2}(f^{(5)} \boxtimes f^{(1)} - S)$. Thus, as there are no non-trivial gauge auto-equivalences of the category $Z(\mathrm{Ad}(E_6^+))$, we must have that $\mathrm{Aut}^{\mathrm{br}}(Z(\mathrm{Ad}(E_6^+))) \subseteq \mathbb{Z}/2\mathbb{Z}$. A non-trivial braided auto-equivalence of $Z(\mathrm{Ad}(E_6^+))$ is constructed in Lemma 3.2.5. □

Remark 3.2.12. The dimensions and T matrix for the centre for $Z(\mathrm{Ad}(E_6^+))$ have also been computed in [30].

We end this section with the most difficult case, the centre of $\mathrm{Ad}(A_{2N+1})$. Unfortunately we are unable to explicitly construct the braided auto-equivalences for many cases. Instead we are forced to find invertible bimodules over $\mathrm{Ad}(A_{2N+1})$ and appeal to the isomorphism from the invertible bimodules to the braided auto-equivalences of the centre.

Lemma 3.2.13. We have

$$\operatorname{Aut}^{\mathrm{br}}(Z(\operatorname{Ad}(A_{2N+1}))) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{when } N \equiv 0 \pmod 2 \text{ or } N = 1 \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{when } N \equiv 1 \pmod 2 \text{ and } N \neq \{1,3\} \\ D_{2\cdot 4} & \text{when } N = 3. \end{cases}$$

Proof. As in the $Z(\operatorname{Ad}(E_6^+))$ case we choose representatives for the simple objects of $Z(\operatorname{Ad}(A_{2N+1}))$. These representatives are:

$$\left\{f^{(n)} \boxtimes f^{(m)} \mid n < 2N - m \text{ and } n - m \equiv 0 \pmod 2\right\} \bigcup \left\{f^{(2N-n)} \boxtimes f^{(n)} \mid 0 \leq n < N\right\} \bigcup \left\{\frac{f^{(N)} \boxtimes f^{(N)} \pm S}{2}\right\}.$$

Thus the dimensions of the simple objects are among the set

$$\{[n+1][m+1] \mid n < 2N - m \text{ and } n - m \equiv 0 \pmod 2\} \bigcup \{[2N-n+1][n+1] \mid 0 \leq n < N\} \bigcup \left\{\frac{[N+1]^2}{2}\right\}$$

When $N \neq \{1,3\}$ the automorphism group of the fusion ring is $(\mathbb{Z}/2\mathbb{Z})^3$, generated by the three automorphisms

- $f^{(n)} \boxtimes f^{(m)} \leftrightarrow f^{(m)} \boxtimes f^{(n)}$ for all n, m even,
- $f^{(n)} \boxtimes f^{(m)} \leftrightarrow f^{(2N-n)} \boxtimes f^{(m)}$ for all n, m odd,
- $\frac{f^{(N)} \boxtimes f^{(N)} + S}{2} \leftrightarrow \frac{f^{(N)} \boxtimes f^{(N)} - S}{2}$.

When $N = 1$ the automorphism group of the fusion ring is S_3, generated by the two automorphisms

- $\frac{f^{(N)} \boxtimes f^{(N)} + S}{2} \leftrightarrow \frac{f^{(N)} \boxtimes f^{(N)} - S}{2}$,
- $f^{(2)} \boxtimes f^{(0)} \to \frac{f^{(N)} \boxtimes f^{(N)} + S}{2} \to \frac{f^{(N)} \boxtimes f^{(N)} - S}{2} \to f^{(2)} \boxtimes f^{(0)}$.

When $N = 3$ the automorphism group of the fusion ring has order 16, and is generated by the three automorphisms

- $\frac{f^{(3)} \boxtimes f^{(3)} + S}{2} \leftrightarrow \frac{f^{(3)} \boxtimes f^{(3)} - S}{2}$,
- $f^{(1)} \boxtimes f^{(1)} \to \frac{f^{(3)} \boxtimes f^{(3)} + S}{2} \to f^{(5)} \boxtimes f^{(1)} \to \frac{f^{(3)} \boxtimes f^{(3)} - S}{2} \to f^{(1)} \boxtimes f^{(1)}$, $f^{(2)} \boxtimes f^{(0)} \leftrightarrow f^{(0)} \boxtimes f^{(4)}$, and $f^{(4)} \boxtimes f^{(0)} \leftrightarrow f^{(0)} \boxtimes f^{(2)}$,
- $f^{(2)} \boxtimes f^{(0)} \leftrightarrow f^{(0)} \boxtimes f^{(2)}$, $f^{(4)} \boxtimes f^{(0)} \leftrightarrow f^{(0)} \boxtimes f^{(4)}$.

It can be easily verified that the first two of these automorphisms satisfy the relations of the usual s and r generators of $D_{2\cdot 4}$.

From Lemma 3.2.4 we know there are no non-trivial braided Gauge auto-equivalences of $Z(\text{Ad}(A_{2N+1}))$. Thus these groups are upper bounds for $\text{Aut}^{\text{br}}(Z(\text{Ad}(A_{2N+1})))$.

Case $N \equiv 0 \pmod 2$: When $N \equiv 0 \pmod 2$ the t values for $f^{(1)} \boxtimes f^{(1)}$ and $f^{(2N-1)} \boxtimes f^{(1)}$ are different, and the t values for $f^{(2)} \boxtimes f^{(0)}$ and $f^{(0)} \boxtimes f^{(2)}$ are different. Thus the only possible braided auto-equivalence is $\frac{f^{(N)} \boxtimes f^{(N)} + S}{2} \leftrightarrow \frac{f^{(N)} \boxtimes f^{(N)} - S}{2}$, which is constructed in Lemma 3.2.4. Thus $\text{Aut}^{\text{br}}(Z(\text{Ad}(A_{2N+1}))) = \mathbb{Z}/2\mathbb{Z}$.

Case $N \equiv 1 \pmod 2$ and $N \neq \{1,3\}$: When $N \equiv 1 \pmod 2$ and $N \neq \{1\}$ the t-values for $f^{(2)} \boxtimes f^{(0)}$ and $f^{(0)} \boxtimes f^{(2)}$ are different. Thus we have an upper bound of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\text{Aut}^{\text{br}}(Z(\text{Ad}(A_{2N+1})))$.

To complete the proof we need to construct four braided auto-equivalences. Instead we show the existence of four invertible bimodules over $\text{Ad}(A_{2N+1})$. There is the trivial bimodule, of rank $N+1$, which gives us one. The odd graded piece of the $\mathbb{Z}/2\mathbb{Z}$-graded fusion category A_{2N+1} is an invertible bimodule. This bimodule has rank N, and thus is not equivalent to the trivial bimodule. This gives us two invertible bimodules over $\text{Ad}(A_{2N+1})$, if we can show the existence of a third invertible bimodule then the group must be $(\mathbb{Z}/2\mathbb{Z})^2$ via an application of Lagrange's theorem.

Consider the object $A = \mathbf{1} \oplus f^{(2N)}$ in $\text{Ad}(A_{2N+1})$. This object has a unique algebra structure in A_{2N+1} [45, 15], and furthermore the category of A bimodules in A_{2N+1} is equivalent to A_{2N+1} [47]. Therefore we can apply Lemma 2.0.1 to see that $A-\text{bimod}_{\text{Ad}(A_{2N+1})} \simeq \text{Ad}(A_{2N+1})$. Thus A−mod is an invertible bimodule over $\text{Ad}(A_{2N+1})$. The rank of A−mod is $\frac{N+1}{2}$, and so is non-equivalent to either of the two previous invertible bimodules.

Case $N = 1$:

The representatives of the four simple objects of $Z(\text{Ad}(A_3))$ are $f^{(0)} \boxtimes f^{(0)}$, $\frac{f^{(1)} \boxtimes f^{(1)} - S}{2}$, $\frac{f^{(1)} \boxtimes f^{(1)} + S}{2}$, and $f^{(2)} \boxtimes f^{(0)}$. The t-values of these objects are 1, 1, 1, and −1 respectively. We can see that the only possibility for a non-trivial braided auto-equivalence is $\frac{f^{(1)} \boxtimes f^{(1)} + S}{2} \leftrightarrow \frac{f^{(1)} \boxtimes f^{(1)} - S}{2}$. This braided auto-equivalence is constructed in Lemma 3.2.4.

Case $N = 3$: The t-values for $f^{(2)} \boxtimes f^{(0)} \leftrightarrow f^{(0)} \boxtimes f^{(2)}$ are different in $Z(\text{Ad}(A_7))$, thus we have that $\text{Aut}^{\text{br}}(Z(\text{Ad}(A_7))) \subseteq D_{2\cdot 4}$.

As in the $N \equiv 1 \pmod 2$ case we unfortunately have to explicitly construct 8 invertible bimodules over $\text{Ad}(A_7)$. Using the same arguments as before we have the algebra objects $\mathbf{1}$, $\mathbf{1}\oplus f^{(2)}$, and $\mathbf{1}\oplus f^{(6)}$ which give rise to invertible bimodules. Each of these bimodules is distinct as the ranks are 4, 3, and 2 respectively. We can twist each of these bimodules by the non-trivial outer auto-equivalence of $\mathrm{Ad}(A_7)$ from Lemma 3.2.10 to find 6 distinct invertible bimodules over $\mathrm{Ad}(A_7)$. As 6 doesn't divide the order of $D_{2\cdot 4}$ it follows from Lagrange's theorem that $\mathrm{Aut}^{\mathrm{br}}(Z(\mathrm{Ad}(A_7))) = D_{2\cdot 4}$. □

We also need to calculate the braided auto-equivalences of the opposite braided category for many of the examples above. However there is a canonical isomorphism $\mathrm{Aut}^{\mathrm{br}}(C) \cong \mathrm{Aut}^{\mathrm{br}}(C^{\mathrm{bop}})$ so we really don't need to worry.

3.3 The Brauer-Picard groups of the ADE fusion categories

We spend this Section tying up the loose ends to complete our computations of the Brauer-Picard groups of the ADE fusion categories. Our only remaining problem is to compute the braided auto-equivalence group of the centres that are products of two modular categories.

Consider a product of braided tensor categories $C \boxtimes D$. Given a braided auto-equivalence of C and a braided auto-equivalence of D one gets a braided auto-equivalence of $C \boxtimes D$ by acting independently on each factor. This determines an injection $\mathrm{Aut}^{\mathrm{br}}(C) \times \mathrm{Aut}^{\mathrm{br}}(D) \to \mathrm{Aut}^{\mathrm{br}}(C \boxtimes D)$. This injection is an isomorphism if and only if every braided auto-equivalence of $C \boxtimes D$ restricts to braided auto-equivalences of the factors.

For all of the centres we are interested in it turns out that every braided auto-equivalence of the product restricts to auto-equivalences of the factors. Thus the results of Section 3.2 are sufficient to compute the Brauer-Picard group. We consider the dimensions and t-values of the generating objects of the factors (in our examples the factors are always singly generated). These values are invariant under action by a braided auto-equivalence. Thus if the only other objects in the product with the same dimension and t-value as the generating object also lie in the same factor then the generating object of that factor must be mapped within the factor by any braided auto-equivalence. As the generating object of the factor is mapped within the same factor it follows that the rest of the factor must be mapped within the same factor and thus the braided auto-equivalence restricts to that factor.

Lemma 3.3.1. We have $\mathrm{Aut}^{\mathrm{br}}(Z(\mathrm{Ad}(E_8^+))) = \mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(D_{16})) \times \mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(A_4)^{\mathrm{bop}})$.

Proof. Recall from Section 3.1 that the Drinfeld centre of $\mathrm{Ad}(E_8^+)$ is $\mathrm{Ad}(D_{16}^+) \boxtimes \mathrm{Ad}(A_4)^{\mathrm{bop}}$.

Let's first look at $f^{(2)} \boxtimes f^{(0)}$, the generating object of the $\mathrm{Ad}(D_{16}^+)$ factor. This object has unique dimension in $\mathrm{Ad}(D_{16}^+) \boxtimes \mathrm{Ad}(A_4)^{\mathrm{bop}}$, and thus we must have that $f^{(2)} \boxtimes f^{(0)}$ is sent to itself by any auto-equivalence of $\mathrm{Ad}(D_{16}^+) \boxtimes \mathrm{Ad}(A_4)^{\mathrm{bop}}$. As $f^{(2)} \boxtimes f^{(0)}$ tensor generates the $\mathrm{Ad}(D_{16}^+)$ factor, we see that any auto-equivalence of $\mathrm{Ad}(D_{16}^+) \boxtimes \mathrm{Ad}(A_4)^{\mathrm{bop}}$ restricts to this factor.

Now we look at the generating object of the $\mathrm{Ad}(A_4)^{\mathrm{bop}}$ factor, that is $f^{(0)} \boxtimes f^{(2)}$. Again by considering dimensions we see that the object $f^{(0)} \boxtimes f^{(2)}$ must be sent to itself by any auto-equivalence. Hence any auto-equivalence of $D_{16}^+ \boxtimes \mathrm{Ad}(A_4)^{\mathrm{bop}}$ restricts to the $\mathrm{Ad}(A_4)^{\mathrm{bop}}$ factor. □

The rest of the Drinfeld centres described in Section 3.1 which are products have braided auto-equivalence groups which decompose in a similar fashion.

Lemma 3.3.2. We have the following:

$$\mathrm{Aut}^{\mathrm{br}}(Z(\mathrm{Ad}(A_{2N}))) = \mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(A_{2N})) \times \mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(A_{2N})^{\mathrm{bop}})$$
$$\mathrm{Aut}^{\mathrm{br}}(Z(\mathrm{Ad}(D_{2N}))) = \mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(D_{2N})) \times \mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(D_{2N})^{\mathrm{bop}}).$$

Proof. In both cases, the pair of the dimension and twist of the generating objects of each factor, $f^{(2)} \boxtimes f^{(0)}$ and $f^{(0)} \boxtimes f^{(2)}$, are unique. Thus every braided auto-equivalence of the centres restricts to braided auto-equivalences of the factors. □

Putting everything together we can now complete the proof of the main Theorem of this Chapter.

Proof of Theorem 3.0.1. The results of Section 3.2 and 3.3 compute the braided auto-equivalence groups of the centres of the unitary fusion categories $\mathrm{Ad}(A_N)$, $\mathrm{Ad}(D_{2N})$, $\mathrm{Ad}(E_6^+)$, and $\mathrm{Ad}(E_8^+)$. The isomorphism $\mathrm{Aut}^{\mathrm{br}}(Z(C)) \cong \mathrm{BrPic}(C)$ gives us the Brauer-Picard groups of these categories. Finally as $\mathrm{Ad}(E_6^-)$ is Galois conjugate to $\mathrm{Ad}(E_6^+)$, and $\mathrm{Ad}(E_8^-)$ is Galois conjugate to $\mathrm{Ad}(E_8^+)$ we can apply [13, Lemma 2.10] to give us the Brauer-Picard groups of $\mathrm{Ad}(E_6^-)$ and $\mathrm{Ad}(E_8^-)$. □

3.4 Explicit constructions of invertible bimodules from braided autoequivalences

To compute the extension theory of a fusion category one ideally wants to have explicit descriptions of the invertible bimodules. So far our analysis has just revealed the group structure of the Brauer-Picard groups. In most cases this is sufficient to completely understand the invertible bimodules over the category e.g. The Brauer-Picard group of $\mathrm{Ad}(E_6)$ is $\mathbb{Z}/2\mathbb{Z}$ which corresponds to the two graded pieces of E_6. Exceptions to this are the $\mathrm{Ad}(D_{10})$ category, which has Brauer-Picard group $S_3 \times S_3$, and the $\mathrm{Ad}(A_7)$ category, which has Brauer-Picard group $D_{2\cdot 4}$. The aim of this Section is to give explicit descriptions of all the invertible bimodules over these categories. We achieve this by chasing through the isomorphism $\mathrm{BrPic}(C) \cong \mathrm{Aut}^{\mathrm{br}}(Z(C))$.

An invertible bimodule over C corresponds to a left C-module M along with an equivalence $C \xrightarrow{\sim} \mathrm{Fun}(M,M)_C$ [17]. This description gives rise to a natural action of $\mathrm{Aut}(C)$ on $\mathrm{BrPic}(C)$ by precomposing a tensor auto-equivalence of C to get another equivalence $C \xrightarrow{\sim} C \xrightarrow{\sim} \mathrm{Fun}(M,M)_C$. If we restrict this action to outer auto-equivalences of C then this action is free and faithful. Therefore up to the action of $\mathrm{Out}_\otimes(C)$, invertible bimodules over C correspond to left C-module categories such that $C \cong \mathrm{Fun}(M,M)_C$. In the language of algebra objects this corresponds to finding all simple algebra objects $A \in C$ such that A-bimod $\cong C$. The isomorphism $\mathrm{BrPic}(C) \cong \mathrm{Aut}^{\mathrm{br}}(Z(C))$ allows us to compute the underlying algebra object A corresponding to an invertible bimodule we get from a braided auto-equivalence of $Z(C)$.

Construction 3.4.1. [17]

Let $F \in \mathrm{Aut}^{\mathrm{br}} Z(C)$. We will construct an algebra $A \in C$ such that $A - \mathrm{mod}_C$ is equivalent to the image of F under the isomorphism 3.0.1 (as left C-module categores). The object $\mathbf{1}$ is trivially an algebra object in C, therefore inducing $\mathbf{1}$ up to the centre of C gives an algebra object of the centre, as the induction functor is lax monoidal. As F is a tensor auto-equivalence $F^{-1}(I(\mathbf{1}))$ also has the structure of an algebra. Finally restricting back down to C gives us an algebra object back in C. However $R(F^{-1}(I(\mathbf{1})))$ may not be indecomposable as an algebra object. Let A be any simple algebra object in the decomposition of $R(F^{-1}(I(\mathbf{1})))$, then independent of choice of A the corresponding module category is always the same. Thus we can choose any such A.

Invertible bimodules over $\mathrm{Ad}(D_{10})$

The issue with directly applying the above construction to get such an A is that it can be difficult to determine how $R(F^{-1}(I(\mathbf{1})))$ decomposes into simple algebra objects. To deal with this problem we use the algorithm described in [24, Chapter 3] to obtain a finite list of possible simple algebra objects in $\mathrm{Ad}(D_{10})$. Note that the algorithm does not guarantee all of the objects returned can be realised as algebra objects. We include the rank of the corresponding module category as this information will be a useful later.

Proposition 3.4.1. *Let A be an algebra object of $\mathrm{Ad}(D_{10})$. Then A is one of:*

Rank 3	Rank 4	Rank 6
$\mathbf{1} \oplus f^{(6)}$	$\mathbf{1} \oplus f^{(2)}$	$\mathbf{1}$
$\mathbf{1} \oplus f^{(2)} \oplus f^{(6)} \oplus P \oplus Q$	$\mathbf{1} \oplus P$	$\mathbf{1} \oplus f^{(4)} \oplus P$
$\mathbf{1} \oplus 2f^{(2)} \oplus 3f^{(4)} \oplus 4f^{(6)} \oplus 2P \oplus 2Q$	$\mathbf{1} \oplus Q$	$\mathbf{1} \oplus f^{(4)} \oplus Q$
	$\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus f^{(6)}$	$\mathbf{1} \oplus f^{(2)} \oplus f^{(4)}$
	$\mathbf{1} \oplus f^{(4)} \oplus f^{(6)} \oplus P$	$\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus f^{(6)} \oplus P \oplus Q$
	$\mathbf{1} \oplus f^{(4)} \oplus f^{(6)} \oplus Q$	$\mathbf{1} \oplus f^{(2)} \oplus 2f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q$
	$\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q$	
	$\mathbf{1} \oplus 2f^{(2)} \oplus 2f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q$	
	$\mathbf{1} \oplus f^{(2)} \oplus 2f^{(4)} \oplus 2f^{(6)} \oplus 2P \oplus Q$	
	$\mathbf{1} \oplus f^{(2)} \oplus 2f^{(4)} \oplus 2f^{(6)} \oplus P \oplus 2Q$	

Proof. This list was computed using the algorithm from [24, Chapter 3]. Our implementation was tested against the results of [24]. □

With this list of possible simple algebra objects we wish to determine how $R(F^{-1}(I(\mathbf{1})))$ decomposes into simple algebra objects.

Recall from Lemmas 3.2.7 and 3.3.2 that the group of braided auto-equivalences of $Z(\mathrm{Ad}(D_{10})) = \mathrm{Ad}(D_{10}) \boxtimes \mathrm{Ad}(D_{10})^{\mathrm{bop}}$ is $S_3 \times S_3$ where each S_3 factor independently permutes the objects $f^{(2)}, P$ and Q in $\mathrm{Ad}(D_{10})$ and $\mathrm{Ad}(D_{10})^{\mathrm{bop}}$ respectively.

In general describing the induction and restriction functors between the categories C and $Z(C)$ is a difficult problem. However as $\mathrm{Ad}(D_{10})$ is modular these functors behave quite nicely. The induction of the tensor identity $\mathbf{1}$ up to the centre $\mathrm{Ad}(D_{10}) \boxtimes \mathrm{Ad}(D_{10})^{\mathrm{bop}}$ gives the object $\oplus_{X \in \mathrm{Irr}(\mathrm{Ad}(D_{10}))} X \boxtimes X^{\mathrm{bop}}$ where we write X^{bop} to specify the object X in the opposite category. The restriction of an object $X \boxtimes Y^{\mathrm{bop}}$ in $\mathrm{Ad}(D_{10}) \boxtimes \mathrm{Ad}(D_{10})^{\mathrm{bop}}$ back down to $\mathrm{Ad}(D_{10})$ is given by $X \otimes Y^*$. We compute the following table of $R(F^{-1}(I(\mathbf{1})))$ for each $F \in \mathrm{Aut}^{\mathrm{br}}(Z(D_{10}))$.

	$\times$	$\mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(D_{10}))$					
		id	$P \leftrightarrow Q$	$f^{(2)} \leftrightarrow P$	$f^{(2)} \leftrightarrow Q$	$f^{(2)} \to P \to Q$	$f^{(2)} \to Q \to P$
$\mathrm{Aut}^{\mathrm{br}}(\mathrm{Ad}(D_{10}^{\mathrm{bop}}))$	id	I	$B_{f^{(2)}}$	B_Q	B_P	A	A
	$P \leftrightarrow Q$	$B_{f^{(2)}}$	I	A	A	B_P	B_Q
	$f^{(2)} \leftrightarrow P$	B_Q	A	I	A	$B_{f^{(2)}}$	B_P
	$f^{(2)} \leftrightarrow Q$	B_P	A	A	I	B_Q	$B_{f^{(2)}}$
	$f^{(2)} \to P \to Q$	A	B_P	$B_{f^{(2)}}$	B_Q	I	A
	$f^{(2)} \to Q \to P$	A	B_Q	B_P	$B_{f^{(2)}}$	A	I

Where

$$\begin{aligned}
I &\coloneqq 6\mathbf{1} \oplus 3f^{(2)} \oplus 6f^{(4)} \oplus 3f^{(6)} \oplus 3P \oplus 3Q \\
A &\coloneqq 3\mathbf{1} \oplus 3f^{(2)} \oplus 3f^{(4)} \oplus 6f^{(6)} \oplus 3P \oplus 3Q \\
B_{f^{(2)}} &\coloneqq 4\mathbf{1} \oplus 5f^{(2)} \oplus 4f^{(4)} \oplus 5f^{(6)} \oplus 2P \oplus 2Q \\
B_P &\coloneqq 4\mathbf{1} \oplus 2f^{(2)} \oplus 4f^{(4)} \oplus 5f^{(6)} \oplus 5P \oplus 2Q \\
B_Q &\coloneqq 4\mathbf{1} \oplus 2f^{(2)} \oplus 4f^{(4)} \oplus 5f^{(6)} \oplus 2P \oplus 5Q.
\end{aligned}$$

Lemma 3.4.2. There exist unique decompositions of I, $B_{f^{(2)}}$,B_P, and B_Q into simple algebra objects. These decompositions are

$$\begin{aligned}
I =&(\mathbf{1}) \oplus (\mathbf{1} \oplus f^{(4)} \oplus P) \oplus (\mathbf{1} \oplus f^{(4)} \oplus Q) \oplus (\mathbf{1} \oplus f^{(2)} \oplus f^{(4)}) \oplus (\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus f^{(6)} \oplus P \oplus Q) \\
&\oplus (\mathbf{1} \oplus f^{(2)} \oplus 2f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q), \\
B_{f^{(2)}} =&(\mathbf{1} \oplus f^{(2)}) \oplus (\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus f^{(6)}) \oplus (\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q) \\
&\oplus (\mathbf{1} \oplus 2f^{(2)} \oplus 2f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q), \\
B_P =&(\mathbf{1} \oplus P) \oplus (\mathbf{1} \oplus f^{(4)} \oplus f^{(6)} \oplus P) \oplus (\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q) \\
&\oplus (\mathbf{1} \oplus f^{(2)} \oplus 2f^{(4)} \oplus 2f^{(6)} \oplus 2P \oplus Q), \\
B_Q =&(\mathbf{1} \oplus Q) \oplus (\mathbf{1} \oplus f^{(4)} \oplus f^{(6)} \oplus Q) \oplus (\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q) \\
&\oplus (\mathbf{1} \oplus f^{(2)} \oplus 2f^{(4)} \oplus 2f^{(6)} \oplus P \oplus 2Q).
\end{aligned}$$

Proof. We brute force check all possible combinations of simple algebra objects in Lemma 3.4.1 and see that only the above decompositions are possible. To simplify our computations recall that all simple algebra objects in the decomposition of $R(F^{-1}(I(\mathbf{1})))$ will have equivalent module categories. As module categories with different ranks are clearly non-equivalent we can restrict our attention to combinations of algebra objects whose corresponding module categories have the same rank. $\square$

Lemma 3.4.3. The objects $\mathbf{1} \oplus f^{(2)}$, $\mathbf{1} \oplus P$, and $\mathbf{1} \oplus Q$ have unique algebra object structures.

Proof. Existence of an algebra object structure follows from Lemma 3.4.2. As $\mathrm{Hom}(f^{(2)} \otimes f^{(2)}, f^{(2)})$, $\mathrm{Hom}(P \otimes P, P)$, and $\mathrm{Hom}(Q \otimes Q, Q)$ are all 1-dimensional we can apply [45, Lemma 8] to get uniqueness. □

Lemma 3.4.4. The algebra object A decomposes in to simple algebra objects as $(\mathbf{1} \oplus f^{(6)}) \oplus (\mathbf{1} \oplus f^{(2)} \oplus f^{(6)} \oplus P \oplus Q) \oplus (\mathbf{1} \oplus 2f^{(2)} \oplus 3f^{(4)} \oplus 4f^{(6)} \oplus 2P \oplus 2Q)$.

Proof. Using the same technique as in Lemma 3.4.2 we can show that A either decomposes as

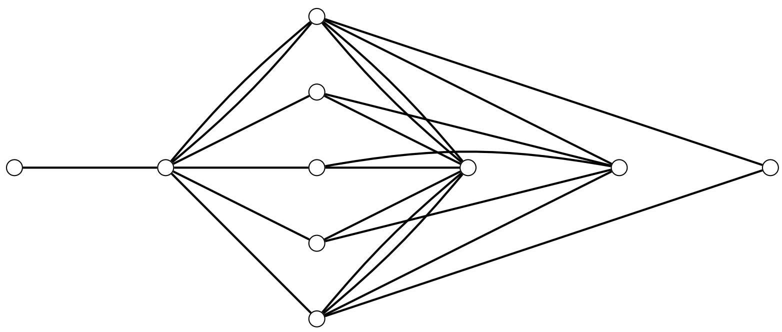
$$(\mathbf{1} \oplus f^{(6)}) \oplus (\mathbf{1} \oplus f^{(2)} \oplus f^{(6)} \oplus P \oplus Q) \oplus (\mathbf{1} \oplus 2f^{(2)} \oplus 3f^{(4)} \oplus 4f^{(6)} \oplus 2P \oplus 2Q),$$

or

$$3 \times (\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q).$$

We will rule out the possibility of the latter decomposition.

Suppose such a decomposition did exist, then this would imply that the corresponding bimodule category would have underlying algebra object $\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q$, and hence is equivalent to $\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q$-mod as a left module category. The category $\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q$-mod has module fusion graph:



The dimension of object at the far right we compute to be $\sqrt{[3]+1}$. Thus the internal hom of this object is an algebra object in $\mathrm{Ad}(D_{10})$ with dimension $[3]+1$, and so must be one of $\mathbf{1} \oplus f^{(2)}$, $\mathbf{1} \oplus P$, or $\mathbf{1} \oplus Q$. This implies $\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q$-mod is equivalent to one of $\mathbf{1} \oplus f^{(2)}$-mod, $\mathbf{1} \oplus P$-mod, or $\mathbf{1} \oplus Q$-mod.

Assume without loss of generality that $\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q$-mod is equivalent to $\mathbf{1} \oplus f^{(2)}$-mod, as the following argument works with $f^{(2)}$ replaced with P or Q. Lemma 3.4.3 shows us that there exists a unique algebra object structure on $\mathbf{1} \oplus f^{(2)}$, and thus up to action of tensor auto-equivalences there is a unique invertible bimodule with underlying algebra object $\mathbf{1} \oplus f^{(2)}$. However

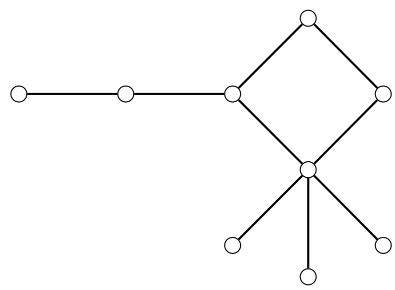
Lemma 3.4.2 tells us that the underlying algebra object of any of the invertible bimodules corresponding to $B_{f^{(2)}}$ must be $\mathbf{1} \oplus f^{(2)}$. As $B_{f^{(2)}}$ appears 6 times in Table 3.4 it follows that the algebra object structure on $\mathbf{1} \oplus f^{(2)}$ coming from A is different than the one coming from $B_{f^{(2)}}$. Thus there are two algebra object structures on $\mathbf{1} \oplus f^{(2)}$, but this is a contradiction to Lemma 3.4.3. Therefore A cannot decompose as $3 \times (\mathbf{1} \oplus f^{(2)} \oplus f^{(4)} \oplus 2f^{(6)} \oplus P \oplus Q)$. □

Corollary 3.4.2. *There exists two algebra object structures on* $\mathbf{1} \oplus f^{(6)}$ *such that* $(\mathbf{1} \oplus f^{(6)}) - \text{bimod} \simeq \text{Ad}(D_{10})$.

Proof. The above table and Lemma 3.4.4 shows that there are exactly 12 invertible bimodule categories over $\text{Ad}(D_{10})$ whose underlying algebra object is $\mathbf{1} \oplus f^{(6)}$. As $\text{Out}_{\otimes}(\text{Ad}(D_{10})) = S_3$ has order 6 (Lemma 3.2.7), up to the action of outer tensor auto-equivalences there are two different left $\text{Ad}(D_{10})$ modules with dual equivalent to $\text{Ad}(D_{10})$, and hence two algebra object structures on $\mathbf{1} \oplus f^{(6)}$ such that $(\mathbf{1} \oplus f^{(6)})$-bimod $\simeq \text{Ad}(D_{10})$. □

The algebra objects $\mathbf{1} \oplus f^{(6)}$ correspond to subfactors. These are the GHJ subfactors corresponding to the odd part of E_7 as a module over D_{10}.

Corollary 3.4.3. *There exist two inequivalent subfactors of index* $1+\cos(\frac{\pi}{9})\csc(\frac{\pi}{18}) \approx 6.411$ *whose even and dual even parts are* $\text{Ad}(D_{10})$*, and with principle graph*



Summarising the above discussion, we have:

Theorem 3.4.5. Up to action of outer tensor auto-equivalences there are six invertible bimodules over $\text{Ad}(D_{10})$. These invertible bimodules come from the algebra objects $\mathbf{1}$, $\mathbf{1} \oplus f^{(2)}$, $\mathbf{1} \oplus P$, and $\mathbf{1} \oplus Q$ each of which have a unique algebra object structure, and $\mathbf{1} \oplus f^{(6)}$ which has two algebra object structures.

Bimodules over $\text{Ad}(A_7)$

Recall that the group of braided auto-equivalences of $Z(\text{Ad}(A_7))$ is $D_{2\cdot 4}$ with generators r and s as described in Lemma 3.2.13. The same calculations as for D_{10} reveal the group structure on the invertible bimodules of $\mathrm{Ad}(A_7)$. The induction matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

for this computation was computed from the induction matrix for the modular category A_7. The ordering of the rows is the standard ordering of the simples objects of $\mathrm{Ad}(A_7)$. The ordering of the columns is as follows:

$$\mathbf{1} \boxtimes \mathbf{1}, f^{(2)} \boxtimes \mathbf{1}, f^{(4)} \boxtimes \mathbf{1}, f^{(6)} \boxtimes \mathbf{1}, f^{(1)} \boxtimes f^{(1)}, f^{(3)} \boxtimes f^{(1)}, f^{(5)} \boxtimes f^{(1)}, \mathbf{1} \boxtimes f^{(2)}, f^{(2)} \boxtimes f^{(2)},$$
$$f^{(4)} \boxtimes f^{(2)}, f^{(1)} \boxtimes f^{(3)}, \frac{f^{(3)} \boxtimes f^{(3)} + S}{2}, \frac{f^{(3)} \boxtimes f^{(3)} - S}{2}, \mathbf{1} \boxtimes f^{(4)}.$$

With this data we compute the group structure as follows:

$\mathrm{Aut}^{\mathrm{br}}(Z(\mathrm{Ad}(A_7)))$	e	r	r^2	r^3	s	rs	r^2s	r^3s
A	$\mathbf{1}$	$\mathbf{1} \oplus f^{(2)}$	$\mathbf{1} \oplus f^{(6)}$	$\mathbf{1} \oplus f^{(4)}$	$\mathbf{1} \oplus f^{(2)}$	$\mathbf{1}$	$\mathbf{1} \oplus f^{(4)}$	$\mathbf{1} \oplus f^{(6)}$

Chapter 4

Classifying cyclic extensions of the adjoint subcategories of the ADE fusion categories generated by an object of dimension less than 2

With the computations of Chapter 3 we are now placed to attempt the classification of unitary fusion categories generated by a normal object of dimension less than 2. Unfortunately we are unable to provide a complete classification, as there is a gap when the dimension is $\sqrt{2+\sqrt{2}}$. The obstruction to classifying at this dimension is the possible existence of an interesting new rank 12 extension of $\mathrm{Ad}(A_7)$. We are unable to construct, or show non-existence, for such a category, though we have some evidence that the category should exist. Supposing the category did exist, then we are able to work out what the fusion rules would be. At the end of the Chapter we present the fusion graph for the generating normal object of dimension $\sqrt{2+\sqrt{2}}$.

4.1 Fusion categories generated by an object of dimension 1

We begin this Chapter by reproving a well known classification, the classification of unitary fusion categories generated by an object of dimension 1. While this classification is certainly well known, we give our version of the proof, as it is a nice warm up for the more difficult classification problems we will deal with in this Chapter. This result is also interesting to us because the categories appearing in this classification will appear in various contexts throughout the classification results of the rest of this Chapter.

Theorem 4.1.1. Let C be a unitary fusion category generated by an object X with dimension 1. Then

$$C \simeq \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

where $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$.

Proof. Let C be such a category, then as X tensor generates and has dimension 1, every simple object in C must also have dimension 1. This implies $\mathrm{Ad}(C) \simeq \mathsf{Vec}$ and thus C is a G-graded extension of Vec. As C is generated by a single object, we must have that $G \cong \mathbb{Z}/M\mathbb{Z}$ for some $M \in \mathbb{N}$. Hence the proof of this Theorem reduces to classifying cyclic extensions of Vec (all of which are unitary as Vec is completely unitary).

As there are no non-trivial invertible objects in Vec, we have that

$$H^2(\mathbb{Z}/M\mathbb{Z}, \mathrm{Inv}(\mathsf{Vec})) = \{e\},$$

and it is well known that $H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times) \cong \mathbb{Z}/M\mathbb{Z}$. Thus there are at most M possible $\mathbb{Z}/M\mathbb{Z}$-graded extensions of Vec. These are all realised by the categories $\mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z})$ for $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$. □

Note that the above Theorem does not give the classification of such categories up to monoidal equivalence, but merely up to equivalence as extensions of Vec.

4.2 Realising bimodule categories, and additional information

In this section we summarise the results of Chapter 3 needed to compute the cyclic extension theory of the adjoint subcategories of the ADE fusion categories. We give explicit descriptions of the invertible bimodules over the adjoint subcategories of the ADE fusion categories, along with the order of each bimodule. This information will be useful when we try to classify cyclic homomorphisms into the Brauer-Picard groups.

Let M be an invertible bimodule over a $\mathbb{Z}/2\mathbb{Z}$-graded fusion category C. Then M splits into two invertible bimodule categories over $\mathrm{Ad}(C)$. We call these two $\mathrm{Ad}(C)$ bimdoules, M^{even} and M^{odd}. We can realise all the bimodules over the adjoint subcategories of the ADE fusion categories as even and odd parts of bimodules over the full ADE fusion categories, along with twistings by auto-equivalences as in Definition 2.0.9. Over the A_N fusion categories we have the trivial A_N bimodule for all N, and the $D_{\frac{N+3}{2}}$ bimodule when $N \equiv 3 \pmod 4$ (we also have the D_{even} modules when $N \equiv 3 \pmod 4$, but these don't have the structure of invertible bimodules over A_N). The auto-equivalences of $\mathrm{Ad}(A_N)$ are trivial, except for when $N = 7$, in which case there is a single non-trivial auto-equivalence sending $f^{(2)} \leftrightarrow f^{(4)}$. Over the D_{2N} fusion categories we have the trivial D_{2N} bimodule for all N, and the E_7, and $\overline{E_7}$ bimodules when $N = 5$ (both E_7 and $\overline{E_7}$ have the same bimodule fusion rules). There is always an order two auto-equivalence of $\mathrm{Ad}(D_{2N})$ that sends $P \leftrightarrow Q$, and when $N = 5$ there is an order 3 auto-equivalence sending $f^{(2)} \mapsto P \mapsto Q \mapsto f^{(2)}$. Over both E_6 and E_8 there is just the trivial bimodule, and no non-trivial auto-equivalences of the adjoint subcategories. We summarise this information in Table 4.1. In particular we see that every bimodule is unitary, and thus the adjoint subcategories of the ADE fusion categories are completely unitary. Thus every graded extension of these categories is again unitary.

We also present Table 4.2 showing the invertible objects in the centres of each adjoint subcategory of an ADE fusion category. Here we use the notation that these invertible objects are idempotents in the planar algebra corresponding to the centre, described in Chapter 3. When an invertible bimodule acts non-trivially on the group of invertible objects in the centre, then we also include this information. This information will help us to determine the number of possible cyclic extensions of each category.

C		Bimodule categories over C	Orders
$\mathrm{Ad}(A_N)$	$N = 3$	A_3^{even} and A_3^{odd}	$1, 2$
	$N = 7$	A_7^{even}, A_7^{odd}, D_5^{even}, D_5^{odd}, ${}_{f^{(2)}\leftrightarrow f^{(4)}}A_7^{\text{even}}$, ${}_{f^{(2)}\leftrightarrow f^{(4)}}A_7^{\text{odd}}$, ${}_{f^{(2)}\leftrightarrow f^{(4)}}D_5^{\text{even}}$, and ${}_{f^{(2)}\leftrightarrow f^{(4)}}D_5^{\text{odd}}$	$1, 2, 2, 2, 2, 4, 4, 2$
	$N \equiv 0 \pmod 2$	A_N^{even}	1
	$N \equiv 1 \pmod 4$	A_N^{even} and A_N^{odd}	$1, 2$
	$N \equiv 3 \pmod 4$ and $N \neq \{3, 7\}$	A_N^{even}, A_N^{odd}, $D_{\frac{N+1}{2}+1}^{\text{even}}$, and $D_{\frac{N+1}{2}+1}^{\text{odd}}$	$1, 2, 2, 2$
$\mathrm{Ad}(D_{2N})$	$N \neq 5$	D_{2N}^{even}, D_{2N}^{odd}, ${}_{P\leftrightarrow Q}D_{2N}^{\text{even}}$, and ${}_{P\leftrightarrow Q}D_{2N}^{\text{odd}}$	$1, 2, 2, 2$
	$N = 5$	D_{10}^{even}, D_{10}^{odd}, E_7^{even}, $\overline{E_7}^{\text{even}}$, E_7^{odd}, $\overline{E_7}^{\text{odd}}$,	$1, 2, 2, 2, 3, 3$
		${}_{P\leftrightarrow Q}D_{10}^{\text{odd}}$, ${}_{P\leftrightarrow Q}D_{10}^{\text{even}}$, ${}_{P\leftrightarrow Q}E_7^{\text{odd}}$, ${}_{P\leftrightarrow Q}\overline{E_7}^{\text{odd}}$, ${}_{P\leftrightarrow Q}\overline{E_7}^{\text{even}}$, ${}_{P\leftrightarrow Q}E_7^{\text{even}}$,	$2, 2, 2, 2, 6, 6$
		${}_{f^{(2)}\leftrightarrow P}E_7^{\text{even}}$, ${}_{f^{(2)}\leftrightarrow P}E_7^{\text{odd}}$, ${}_{f^{(2)}\leftrightarrow P}D_{10}^{\text{even}}$, ${}_{f^{(2)}\leftrightarrow P}\overline{E_7}^{\text{odd}}$, ${}_{f^{(2)}\leftrightarrow P}D_{10}^{\text{odd}}$, ${}_{f^{(2)}\leftrightarrow P}\overline{E_7}^{\text{even}}$,	$2, 2, 2, 2, 6, 6$
		${}_{f^{(2)}\leftrightarrow Q}\overline{E_7}^{\text{even}}$, ${}_{f^{(2)}\leftrightarrow Q}E_7^{\text{odd}}$, ${}_{f^{(2)}\leftrightarrow Q}\overline{E_7}^{\text{odd}}$, ${}_{f^{(2)}\leftrightarrow Q}D_{10}^{\text{even}}$, ${}_{f^{(2)}\leftrightarrow Q}E_7^{\text{even}}$, ${}_{f^{(2)}\leftrightarrow Q}D_{10}^{\text{odd}}$,	$2, 2, 2, 2, 6, 6$
		${}_{f^{(2)}\mapsto P\mapsto Q}E_7^{\text{odd}}$, ${}_{f^{(2)}\mapsto P\mapsto Q}\overline{E_7}^{\text{even}}$, ${}_{f^{(2)}\mapsto P\mapsto Q}D_{10}^{\text{odd}}$, ${}_{f^{(2)}\mapsto P\mapsto Q}E_7^{\text{even}}$, ${}_{f^{(2)}\mapsto P\mapsto Q}D_{10}^{\text{even}}$, ${}_{f^{(2)}\mapsto P\mapsto Q}\overline{E_7}^{\text{odd}}$,	$3, 6, 6, 6, 3, 3$
		${}_{f^{(2)}\mapsto Q\mapsto P}\overline{E_7}^{\text{odd}}$, ${}_{f^{(2)}\mapsto Q\mapsto P}E_7^{\text{even}}$, ${}_{f^{(2)}\mapsto Q\mapsto P}\overline{E_7}^{\text{even}}$, ${}_{f^{(2)}\mapsto Q\mapsto P}D_{10}^{\text{odd}}$, ${}_{f^{(2)}\mapsto Q\mapsto P}E_7^{\text{odd}}$, ${}_{f^{(2)}\mapsto Q\mapsto P}D_{10}^{\text{even}}$	$3, 6, 6, 6, 3, 3$
$\mathrm{Ad}(E_6^{\pm})$		E_6^{even} and E_6^{odd}	$1, 2$
$\mathrm{Ad}(E_8^{\pm})$		E_8^{even} and E_8^{odd}	$1, 2$

Table 4.1: Bimodules over the adjoint subcategories of the ADE fusion categories

C		$\mathrm{Inv}(Z(C))$	Action of bimodules (when non-trivial)
$\mathrm{Ad}(A_N)$	$N = 3$	$\{\mathbf{1} \boxtimes \mathbf{1}, \frac{f^{(1)} \boxtimes + S}{2}, \frac{f^{(1)} \boxtimes - S}{2}, f^{(2)} \boxtimes \mathbf{1}\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	A_3^{odd} exchanges the objects $\frac{f^{(1)} \boxtimes + S}{2}$ and $\frac{f^{(1)} \boxtimes - S}{2}$
	$N \equiv 0 \pmod 2$	$\{\mathbf{1} \boxtimes \mathbf{1}\}$	
	$N \equiv 1 \pmod 2$	$\{\mathbf{1} \boxtimes \mathbf{1}, f^{(N-1)} \boxtimes \mathbf{1}\}$	
$\mathrm{Ad}(D_{2N})$	$N = 2$	$\{\mathbf{1} \boxtimes \mathbf{1}, \mathbf{1} \boxtimes P,$ $P \boxtimes \mathbf{1}, P \boxtimes P, Q \boxtimes Q,$ $\mathbf{1} \boxtimes Q, Q \boxtimes \mathbf{1}, Q \boxtimes P, Q \boxtimes Q\} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	D_4^{odd} applies $P \leftrightarrow Q$ to the second factor ${}_{P\leftrightarrow Q}D_4^{\mathrm{odd}}$ applies $P \leftrightarrow Q$ to the first factor ${}_{P\leftrightarrow Q}D_4^{\mathrm{even}}$ applies $P \leftrightarrow Q$ to both factors
	$N > 2$	$\{\mathbf{1} \boxtimes \mathbf{1}\}$	
$\mathrm{Ad}(E_6^{\pm})$		$\{\mathbf{1} \boxtimes \mathbf{1}, f^{(10)} \boxtimes \mathbf{1}\}$	
$\mathrm{Ad}(E_8^{\pm})$		$\{\mathbf{1} \boxtimes \mathbf{1}\}$	

Table 4.2: Invertible objects in the centre, and action by interesting bimodules

Recall we only care about extensions generated by an object of dimension less than 2. We compute the dimensions of the objects in each of our bimodule categories, as this will allow us to rule out many extensions that can not be generated by such an object, and thus disqualify certain cyclic homomorphisms into the Brauer-Picard group. As twisting a bimodule by a monoidal auto-equivalence of the underlying category doesn't change the dimensions of the objects, we only include the dimensions of the untwisted bimodules.

Dimensions in the A series

Let $q = e^{\frac{\pi i}{N+1}}$, then the dimensions of the simple objects in the invertible bimodules (when they exist) over $\mathrm{Ad}(A_N)$ are:

Bimodule	Dimensions of simples
A_N^{even}	$\{[2n-1]_q : 1 < n < \lceil \frac{N}{2} \rceil\}$
A_N^{odd}	$\{[2n]_q : 1 < n < \lfloor \frac{N}{2} \rfloor\}$
$D^{\text{even}}_{\frac{N+1}{2}+1}$	$\{\sqrt{2}[2n-1]_q : 1 < n \leq \frac{N+1}{4}\}$
$D^{\text{odd}}_{\frac{N+1}{2}+1}$	$\{\sqrt{2}[2n]_q : 1 < n < \frac{N+1}{4}\} \bigcup \{\sqrt{2}[\frac{N+1}{2}]_q\}$.

Dimensions in the D series

Let $q = e^{\frac{\pi i}{4N-2}}$, then the dimensions of the simple objects in the invertible bimodules over $\mathrm{Ad}(D_{2N})$ (when they exist) are:

Bimodule	Dimensions of simples
D_{2N}^{even}	$\{[2n-1]_q : 1 < n < N\} \bigcup \{\frac{[2N-1]_q}{2}\}$
D_{2N}^{odd}	$\{[2n]_q : 1 < n < N\}$
E_7^{even}	$\cos\left(\frac{\pi}{18}\right)\{2, \mathrm{Root}\left[\#1^3 - 12\#1 - 8\&, 3\right], \mathrm{Root}\left[\#1^3 - 6\#1^2 + 8\&, 3\right], \mathrm{Root}\left[\#1^3 - 6\#1^2 + 24\&, 3\right]\}$ $\approx \{1.96962, 3.70167, 4.98724, 5.67128\}$
E_7^{odd}	$\cos\left(\frac{\pi}{18}\right)\{\mathrm{Root}\left[\#1^6 - 60\#1^4 + 288\#1^2 - 192\&, 6\right], \mathrm{Root}\left[\#1^6 - 24\#1^4 + 144\#1^2 - 192\&, 6\right], 2\mathrm{Root}\left[\#1^6 - 6\#1^4 + 9\#1^2 - 3\&, 5\right]\}$ $\approx \{2.53209, 3.87939, 7.29086\}$.

Dimensions in the E series

The dimensions of the simple objects in the invertible bimodules over $\mathrm{Ad}(E_6^{\pm})$ are:

Bimodule	Dimensions of simples
E_6^{even}	$\{1, 1+\sqrt{3}\}$
E_6^{odd}	$\{2\cos\left(\frac{\pi}{12}\right)\}$.

Note the we have dropped the ± notation for the bimodules in this case.

The dimensions of the simple objects in the invertible bimodules over $\mathrm{Ad}(E_8^{\pm})$ are:

Bimodule	Dimensions of simples
E_8^{even}	$\{1, \frac{1}{4}\left(\sqrt{5}+\sqrt{6\left(\sqrt{5}+5\right)}+3\right), \frac{1}{8}\left(4\sqrt{5}+\sqrt{6\left(\sqrt{5}+5\right)}+\sqrt{30\left(\sqrt{5}+5\right)}+8\right),$ $\frac{1}{2}\left(\sqrt{5}+1\right)\}$ $\approx \{1, 1.61803, 2.9563, 4.78339\}$
E_8^{odd}	$\{\frac{1}{2}\sqrt{\sqrt{5}+\sqrt{6\left(\sqrt{5}+5\right)}+7},$ $\frac{1}{8}\left(\sqrt{5}+\sqrt{6\left(\sqrt{5}+5\right)}-1\right)\sqrt{\sqrt{5}+\sqrt{6\left(\sqrt{5}+5\right)}+7},$ $\frac{1}{16}\sqrt{\sqrt{5}+\sqrt{6\left(\sqrt{5}+5\right)}+7}\left(-2\sqrt{5}-\sqrt{6\left(\sqrt{5}+5\right)}+\sqrt{30\left(\sqrt{5}+5\right)}+6\right),$ $\frac{1}{4}\left(\sqrt{5}+1\right)\sqrt{\sqrt{5}+\sqrt{6\left(\sqrt{5}+5\right)}+7}\}$ $\approx \{1.98904, 2.40487, 3.21834, 3.89116\}$.

Again we have dropped the ± notation for the bimodules in this case.

4.3 Classification results

We are now in place to begin classifying unitary fusion categories generated by a normal object of dimension less than 2. By Theorem 1.0.1 such categories must be unitary cyclic extensions of the even part of an ADE fusion category. As we have shown that the adjoint subcategories of the ADE fusion categories are completely unitary, we have that any extension is monoidally equivalent to a unitary extension. Thus in this section we compute cyclic extensions of the adjoint subcategories of the ADE fusion categories, generated by an object of dimension less than 2.

Our proofs all follow the same outline. First we begin by classifying cyclic homomorphisms into the Brauer-Picard group of each category. Using the results of the previous Section we rule out many of these homomorphisms, simply by the fact that the bimodule corresponding to the 1-graded piece doesn't have an object of the right dimension.

Next we use the classification theory of graded extensions to count an upper bound of possible extensions corresponding to each homomorphism. Recall that $\mathbb{Z}/M\mathbb{Z}$-graded extensions of C are classified by triples (c, W, A), with c being the homomorphism $\mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(C)$, W an element of a certain $H^2(\mathbb{Z}/M\mathbb{Z}, \mathrm{Inv}(Z(C)))$-torsor, and A an element of a certain $H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$-torsor, such that certain obstructions vanish. Once we have fixed our homomorphism c, we can easily give an upper bound on the number of extensions realising c. This upper bound is simply the product of the order of the groups $H^2(\mathbb{Z}/M\mathbb{Z}, \mathrm{Inv}(Z(C)))$ and $H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$. Computing the size of these groups is a straightforward exercise in group cohomology, given the information provided about $\mathrm{Inv}(Z(C))$ and the action of bimodules in Table 4.2.

Finally we construct extensions to realise the upper bound. For the most part constructing these extensions is straightforward, simply involving some constructions with known categories. However in the cases $\mathrm{Ad}(A_3)$, $\mathrm{Ad}(D_4)$, and $\mathrm{Ad}(D_{10})$ we have to use more complicated methods.

Cyclic extensions of $\mathrm{Ad}(A_{2N})$

By far the easiest cases are the categories $\mathrm{Ad}(A_{2N})$. This is due to the fact that the Brauer-Picard group is trivial, and there are no non-trivial invertible objects in the centre. Thus we begin our classifications with this case.

Lemma 4.3.1. *Let C be a $\mathbb{Z}/M\mathbb{Z}$-graded extension of $\mathrm{Ad}(A_{2N})$. If C is generated by an object of dimension less than 2, then*

$$C \simeq \mathrm{Ad}(A_{2N}) \boxtimes \mathsf{Vec}^\omega(\mathbb{Z}/M\mathbb{Z}),$$

where $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$.

Proof. We begin by classifying group homomorphisms $\phi : \mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(A_{2N}))$. As the Brauer-Picard group of $\mathrm{Ad}(A_N)$ is trivial the only homomorphism $\mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(A_{2N}))$ is the map $1 \mapsto A_{2N}^{\text{even}}$.

From Table 4.2 we know that $Z(\mathrm{Ad}(A_{2N}))$ has no invertible objects. Therefore $H^2(\mathbb{Z}/M\mathbb{Z}, \mathrm{Inv}(Z(\mathrm{Ad}(A_{2N}))))$ must be trivial for all M. The group $H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$ s just $\mathbb{Z}/M\mathbb{Z}$.

Thus there are at most M different $\mathbb{Z}/M\mathbb{Z}$ graded extensions of $\mathrm{Ad}(A_{2N})$. The categories $\mathrm{Ad}(A_{2N}) \boxtimes \mathsf{Vec}^\omega(\mathbb{Z}/M\mathbb{Z})$ for $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$ realise all M of these extensions. □

Cyclic extensions of $\mathrm{Ad}(A_{2N+1})$, $N \neq \{1,3\}$

To construct some of the extensions in this case we have to take cyclic crossed products (as defined in Definition 2.0.5) of the A_{odd} categories. Recall in general to construct a crossed product we need a monoidal functor $\underline{G} \to \underline{\mathrm{Aut}_\otimes}(C)$. For our specific case we need a monoidal functor $\underline{\mathbb{Z}/M\mathbb{Z}} \to \underline{\mathrm{Aut}_\otimes}(A_{\text{odd}})$ for even M. As A_{odd} is a modular category with distinguished boson or fermion (depending on the exact choice of N) we can apply [14, Theorem 3.3] to get an order two auto-equivalence of A_{odd}. With the explicit description given in the cited Lemma, it is straightforward to show this non-trivial auto-equivalence extends to a monoidal functor $\underline{\mathbb{Z}/2\mathbb{Z}} \to \underline{\mathrm{Aut}_\otimes}(A_{\text{odd}})$. We then factor $\underline{\mathbb{Z}/M\mathbb{Z}}$ through $\underline{\mathbb{Z}/2\mathbb{Z}}$ to get the desired monoidal functor.

Lemma 4.3.2. *Let $N \neq \{1,3\}$ a natural number, and C a $\mathbb{Z}/M\mathbb{Z}$ graded extension of $\mathrm{Ad}(A_{2N+1})$. If C is generated by an object of dimension less than 2, then M is even and*

$$C \simeq \langle (f^{(1)}, 1) \rangle \subset A_{2N+1} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}) \text{ or,}$$
$$C \simeq \langle (f^{(1)}, 1) \rangle \subset A_{2N+1} \overset{\omega}{\rtimes} \mathbb{Z}/M\mathbb{Z},$$

where $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$.

Proof. We begin by classifying group homomorphisms $\phi : \mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(A_{2N+1}))$. Recall that the Brauer-Picard group of $\mathrm{Ad}(A_{2N+1})$ is either $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$ depending on whether N is even or odd.

Case: N even

Here the Brauer-Picard group is $\mathbb{Z}/2\mathbb{Z}$. As the only objects in the trivial bimodule A_{2N+1}^{even} with dimension less than 2 are invertible, we can ignore homomorphisms which map $1 \mapsto A_{2N+1}^{\text{even}}$, as such an object couldn't generate the entire extension. Thus we can assume M even and $\phi : \mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(A_{2N+1}))$ is the map defined by $1 \mapsto {A_{2N+1}}^{\text{odd}}$.

Case: N odd

Here the Brauer-Picard group is $(\mathbb{Z}/2\mathbb{Z})^2$. Exactly as in the N even case we can rule out homomorphisms defined by $1 \mapsto A_{2N+1}^{\text{even}}$. The only time the bimodule D_{N+2}^{odd} contains an object of dimension less than 2 is when $N = 3$, which we have excluded in this Lemma. Thus we can rule out homomorphisms defined by $1 \mapsto D_{N+2}^{\text{odd}}$. The bimodule D_{N+2}^{even} contains a single object of dimension less than 2. However this object always has dimension $\sqrt{2}$, and could only generate the entire extension when $N = 1$. Therefore we can rule out homomorphisms defined by $1 \mapsto D_{N+2}^{\text{even}}$. Hence we can assume M even and $\phi : \mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(A_{2N+1}))$ is the map defined by $1 \mapsto {A_{2N+1}}^{\text{odd}}$.

There are exactly two invertible elements in the centre of $\mathrm{Ad}(A_{2N+1})$. Thus $H^2(\mathbb{Z}/M\mathbb{Z}, \mathrm{Inv}(Z(\mathrm{Ad}(A_{2N+1})))) = \mathbb{Z}/2\mathbb{Z}$. As $\mathbb{Z}/M\mathbb{Z}$ is cyclic we have that $H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times) = \mathbb{Z}/M\mathbb{Z}$. Thus there are at most $2M$ different $\mathbb{Z}/M\mathbb{Z}$ graded extensions of $\mathrm{Ad}(A_{2N+1})$. The categories $\langle (f^{(1)}, 1) \rangle \subset A_{2N+1} \overset{\omega}{\rtimes} \mathbb{Z}/M\mathbb{Z}$ and $\langle (f^{(1)}, 1) \rangle \subset A_{2N+1} \boxtimes \mathsf{Vec}^\omega(\mathbb{Z}/M\mathbb{Z})$ for $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$ realise all of these extensions.

□

Cyclic extensions of $\mathrm{Ad}(D_{2N})$, $N \neq \{2, 5\}$

Recall $D_{2N}^{\pm}$ are the unitary fusion categories obtained from the D_{2N} planar algebra with $q = e^{\frac{\pi i}{4N-2}}$ and rotational eigenvalue of the S generator equal to $\pm i$.

Lemma 4.3.3. *Let $N \neq \{2, 5\}$, and C a $\mathbb{Z}/M\mathbb{Z}$-graded extension of $\mathrm{Ad}(D_{2N})$. If C is generated by an object of dimension less than 2, then M is even and*

$$C \simeq \langle (f^{(1)}, 1) \rangle \subset D_{2N}^{\pm} \boxtimes \mathsf{Vec}^\omega(\mathbb{Z}/M\mathbb{Z}),$$

where $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$.

Proof. Recall that the Brauer-Picard group of $\mathrm{Ad}(D_{2N})$ is $(\mathbb{Z}/2\mathbb{Z})^2$.

Case: $N = 3$

When $N = 3$ the trivial bimodule D_6^{even}, and the twisted trivial bimodule $\mathrm{tw}(D_6^{\text{even}})$ both contain non-trivial objects of dimension $\frac{1+\sqrt{5}}{2}$. However any category generated by such an object couldn't generate all of C. This leaves two homomorphisms $\phi : \mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(D_6))$ to consider, the map defined by $1 \mapsto D_6^{\text{odd}}$ and the map defined by $1 \mapsto \mathrm{tw}(D_6^{\text{odd}})$. In particular we may conclude that M is even.

Case: $N \neq \{2, 3, 5\}$

For these cases the only bimodules over D_{2N} with an object of dimension less than 2 are D_{2N}^{odd} and $\mathrm{tw}(D_{2N}^{\text{odd}})$. This leaves two homomorphisms $\phi : \mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(D_{2N}))$ to consider, the map defined by $1 \mapsto D_{2N}^{\text{odd}}$ and the map defined by $1 \mapsto \mathrm{tw}(D_{2N}^{\text{odd}})$. In particular we may conclude that M is even.

For either case we see that M must be even, and there are two homomorphisms $\phi : \mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(D_{2N}))$ to consider.

When $N > 2$ the centre of $\mathrm{Ad}(D_{2N})$ contains no non-trivial invertible objects, and hence $H^2(\mathbb{Z}/M\mathbb{Z}, \mathrm{Inv}(Z(\mathrm{Ad}(D_{2N})))) = \{e\}$. As $\mathbb{Z}/M\mathbb{Z}$ is cyclic we have that $H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times) = \mathbb{Z}/M\mathbb{Z}$. Thus when M is even there are at most $2M$

possible $\mathbb{Z}/M\mathbb{Z}$ graded extensions of $\mathrm{Ad}(D_{2N})$, and when M is odd there are zero. We construct all of the $2M$-graded extensions in the even case. These are realised by the categories $\langle (f^{(1)}, 1)\rangle \subset D_{2N}^{+} \boxtimes \mathsf{Vec}^{\omega_d}(\mathbb{Z}/M\mathbb{Z})$ and $\langle (f^{(1)}, 1)\rangle \subset D_{2N}^{-} \boxtimes \mathsf{Vec}^{\omega_d}(\mathbb{Z}/M\mathbb{Z})$ for $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$. $\square$

Cyclic extensions of $\mathrm{Ad}(E_6^{\pm})$

Recall $E_6^{\pm}$ are the unitary fusion categories obtained from the E_6 planar algebra with choice of $q = e^{\pm\frac{i\pi}{12}}$.

As in the A_{odd} case, we need to take cyclic crossed products of the categories $E_6^{\pm}$ to construct certain extensions of $\mathrm{Ad}(E_6^{\pm})$. Again we factor $\underline{\mathbb{Z}/M\mathbb{Z}}$ through $\underline{\mathbb{Z}/2\mathbb{Z}}$ (as we will see in the upcoming proof that M must be even). The monoidal functor $\underline{\mathbb{Z}/2\mathbb{Z}} \to \underline{\mathrm{Aut}_\otimes(E_6^{\pm})}$ is obtained through realising $E_6^{\pm}$ as the de-equivariantization $(\mathrm{Ad}(E_6^{\pm}) \boxtimes A_3)//\mathsf{Rep}(\mathbb{Z}/2\mathbb{Z})$.

Lemma 4.3.4. *Let C a $\mathbb{Z}/M\mathbb{Z}$ graded extension of $\mathrm{Ad}(E_6^{\pm})$. If C is generated by a normal object of dimension less than 2, then M is even and*

$$C \simeq \langle (f^{(1)}, 1)\rangle \subset E_6^{\pm} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}) \text{ or,}$$
$$C \simeq \langle (f^{(1)}, 1)\rangle \subset E_6^{\pm} \overset{\omega}{\rtimes} \mathbb{Z}/M\mathbb{Z},$$

where $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$.

Proof. Recall the Brauer-Picard group of $\mathrm{Ad}(E_6^{\pm})$ is $\mathbb{Z}/2\mathbb{Z}$. The only objects in the trivial bimodule E_6^{even} with dimension less than 2 are invertible, and hence can't generate the entire category. Thus we can assume that M is even and $\phi : \mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(E_6^{\pm}))$ is the map defined by $1 \mapsto E_6^{\text{odd}}$.

There are exactly two invertible elements in the centre of $\mathrm{Ad}(E_6^{\pm})$. Thus $H^2(\mathbb{Z}/M\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. As $\mathbb{Z}/M\mathbb{Z}$ is cyclic we have that $H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times) = \mathbb{Z}/M\mathbb{Z}$. Thus when M is odd, there are no extensions satisfying the hypothesis, and when M is even there are at most $2M$. These $2M$ $\mathbb{Z}/M\mathbb{Z}$-graded extensions are all realised by the categories $(f^{(1)}, 1)\rangle \subset E_6^{\pm} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z})$ and $\langle (f^{(1)}, 1)\rangle \subset E_6^{\pm} \overset{\omega}{\rtimes} \mathbb{Z}/M\mathbb{Z}$ for $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$. $\square$

Cyclic extensions of $\mathrm{Ad}(E_8^{\pm})$

Recall $E_8^{\pm}$ are the unitary fusion categories obtained from the E_8 planar algebra with choice of $q = e^{\pm\frac{i\pi}{30}}$.

Lemma 4.3.5. *Let C a $\mathbb{Z}/M\mathbb{Z}$-graded extension of* $\mathrm{Ad}(E_8^{\pm})$*. If C is generated by an object of dimension less than 2, then M is even and*

$$C \simeq \langle (f^{(1)}, 1)\rangle \subset E_8^{\pm} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

where $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^{\times})$.

Proof. Recall that the Brauer-Picard group of $\mathrm{Ad}(E_8^{\pm})$ is $\mathbb{Z}/2\mathbb{Z}$. The only object in the trivial bimodule E_8^{even} with dimension less than 2 has dimension $\frac{1+\sqrt{5}}{2}$. Therefore the category generated by this object would be a cyclic extension of $\mathrm{Ad}(A_4)$, and could not generate the whole category. Thus we can assume M is even and $\phi : \mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(E_8^{\pm}))$ is the map defined by $1 \mapsto E_8^{\text{odd}}$. There are no non-trivial invertible elements in the centre of $\mathrm{Ad}(E_8^{\pm})$. Thus $H^2(\mathbb{Z}/M\mathbb{Z}, \{e\}) = \{e\}$. As $\mathbb{Z}/M\mathbb{Z}$ is cyclic we have that $H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^{\times}) = \mathbb{Z}/M\mathbb{Z}$. Thus we have M possible $\mathbb{Z}/M\mathbb{Z}$ graded extensions of $\mathrm{Ad}(E_8^{\pm})$. These are all realised by the categories $\langle (f^{(1)}, 1)\rangle \subset E_8^{\pm} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z})$ for $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^{\times})$. □

Cyclic extensions of $\mathrm{Ad}(A_3)$

As in our previous arguments for $\mathrm{Ad}(A_{\text{odd}})$, we can restrict our attention to even M, and the homomorphism $\phi : \mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(A_3))$ determined by $1 \mapsto A_3^{\text{odd}}$. Recall that $Z(\mathrm{Ad}(A_3))$ forms a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ group, with the action of A_3^{odd} given by exchanging the objects $\frac{f^1 \pm S}{2}$. We compute the cohomology group $H^2(\mathbb{Z}/M\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ to be $\mathbb{Z}/2\mathbb{Z}$ when 4 divides M, and trivial otherwise. The non-trivial 2−cocycle has representative

$$T(n,m) = \begin{cases} \mathbf{1} \boxtimes \mathbf{1} \text{ if } n+m < M \\ f^{(2)} \boxtimes \mathbf{1} \text{ if } n+m \geq M, \end{cases}$$

constructed from the recipe described in [1]. The existence of this non-trivial cocycle makes the cyclic extensions of $\mathrm{Ad}(A_3)$ more interesting than the previous examples.

The categories $\langle (f^{(1)}, 1)\rangle \subset A_3 \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z})$ for $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^{\times})$ realise M different $\mathbb{Z}/M\mathbb{Z}$-graded extensions of $\mathrm{Ad}(A_3)$. When 4 does not divide M a counting argument as in the previous cases shows that these M categories are all the possible $\mathbb{Z}/M\mathbb{Z}$-graded extensions of $\mathrm{Ad}(A_3)$.

When 4 does divide M, we can twist the tensor product of $\langle (f^{(1)}, 1)\rangle \subset A_3 \boxtimes \mathsf{Vec}(\mathbb{Z}/M\mathbb{Z})$ by the 2-cocycle T to get a new tensor product, i.e. for X_n in the

n-th graded component, and Y_m in the m-th graded component, their twisted multiplication is given by:

$$X_n \otimes_T Y_m := T(n,m) \otimes X_n \otimes Y_m.$$

This gives us a new quasi-monoidal category (we haven't yet shown the associator for this category satisfies the pentagon equation). It is straightforward to see that the simple objects of this category can be labelled by:

$$\{0,1,\cdots,M-1\} \quad \bigcup \quad \{X_i : 1 \leq i \leq M/2\},$$

with commutative fusion rules:

$$\begin{aligned} n \otimes m &= n+m \pmod M \\ n \otimes X_i &= X_{n+i \pmod{\frac{M}{2}}} \\ X_i \otimes X_j &= (i+j-1 \pmod{\frac{M}{2}}) \oplus (i+j-1 \pmod{\frac{M}{2}} + \frac{M}{2}). \end{aligned}$$

As the cohomology group $H^4(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$ is trivial, we have from [14, Lemma 4.5] that there exists some associator for this quasi-monoidal category. In fact as $H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times) = \mathbb{Z}/M\mathbb{Z}$, there are M associators for this quasi-monoidal category, one for each $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$. We call these M different $\mathbb{Z}/M\mathbb{Z}$-graded extensions of $\mathrm{Ad}(A_3)$ the $\mathbb{Z}/2\mathbb{Z}$-generalised Moore-Read categories of type (M,ω), or $\mathrm{GMR}_{\mathbb{Z}/2\mathbb{Z}}(M,\omega)$ for short. When $M = 4$, these categories are the fermionic Moore-Read fusion categories [37], hence the naming convention. We also note that these categories are example of generalized Tambara-Yamagami categories of [37].

Another counting argument shows that when 4 divides M, the only possible $\mathbb{Z}/M\mathbb{Z}$-graded extensions of $\mathrm{Ad}(A_3)$ are the categories $\langle (f^{(1)}, 1)\rangle \subset A_3 \boxtimes \mathsf{Vec}^\omega(\mathbb{Z}/M\mathbb{Z})$ and $\mathrm{GMR}_{\mathbb{Z}/2\mathbb{Z}}(M,\omega)$ for $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$. This proves the following Lemma.

Lemma 4.3.6. *Let C a $\mathbb{Z}/M\mathbb{Z}$ graded extension of $\mathrm{Ad}(A_3)$. If C is generated by an object of dimension less than 2, then M is even and*

$$C \simeq \langle (f^{(1)}, 1)\rangle \subset A_3 \boxtimes \mathsf{Vec}^\omega(\mathbb{Z}/M\mathbb{Z}),$$

or 4 divides M and

$$C \simeq GMR_{\mathbb{Z}/2\mathbb{Z}}(M,\omega),$$

with $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$.

Cyclic extensions of $\mathrm{Ad}(D_4)$

As in the $\mathrm{Ad}(A_3)$ case, there exists interesting cocycles that make the extension theory of $\mathrm{Ad}(D_4)$ interesting.

We can restrict out attention to even M, and the homomorphism $\phi : \mathbb{Z}/M\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(D_4))$ is determined either by

$$1 \mapsto D_4^{\mathrm{odd}}$$

or

$$1 \mapsto_{P\leftrightarrow Q} D_4^{\mathrm{odd}},$$

as every object in D_4^{even} or ${}_{P\leftrightarrow Q}D_4^{\mathrm{even}}$ has dimension 1, and could not generate all of C.

Recall that $Z(\mathrm{Ad}(D_4))$ forms a $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ group, with the action of D_4^{odd} given by

$$\begin{aligned} \mathbf{1} \boxtimes P &\leftrightarrow \mathbf{1} \boxtimes Q, \\ P \boxtimes P &\leftrightarrow P \boxtimes Q, \\ Q \boxtimes P &\leftrightarrow Q \boxtimes Q, \end{aligned}$$

and action of ${}_{P\leftrightarrow Q}D_4^{\mathrm{odd}}$ given by

$$\begin{aligned} P \boxtimes \mathbf{1} &\leftrightarrow Q \boxtimes \mathbf{1}, \\ P \boxtimes P &\leftrightarrow Q \boxtimes P, \\ P \boxtimes Q &\leftrightarrow Q \boxtimes Q. \end{aligned}$$

We now break into two cases, depending on the choice of ϕ.

Case: $1 \mapsto D_4^{\mathbf{odd}}$

We compute the cohomology group $H^2(\mathbb{Z}/M\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z})$ to be $\mathbb{Z}/3\mathbb{Z}$ when 6 divides M, and trivial otherwise. Representatives for the non-trivial cocycles are given by

$$T(n,m) = \begin{cases} \mathbf{1} \boxtimes \mathbf{1} \text{ if } n+m < M \\ P \boxtimes \mathbf{1} \text{ if } n+m \geq M, \end{cases}$$

and

$$V(n,m) = \begin{cases} \mathbf{1} \boxtimes \mathbf{1} \text{ if } n+m < M \\ Q \boxtimes \mathbf{1} \text{ if } n+m \geq M, \end{cases}$$

computed from the recipe of [1].

As in the $\mathrm{Ad}(A_3)$ case we can twist the multiplication of $\langle (f^{(1)},1)\rangle \subset D_4^+ \boxtimes \mathsf{Vec}(\mathbb{Z}/M\mathbb{Z})$ by either of the two 2-cocycles T or V to get two new quasi-monoidal categories. These quasi-monoidal categories both have the same fusion ring. The simple objects are labelled by

$$\left\{0,1,\cdots,\frac{3M}{2}-1\right\}\bigcup\left\{X_i : 1\leq i\leq \frac{M}{2},\right\}$$

and the commutative fusion rules are

$$n\otimes m = n+m \pmod{\frac{3M}{2}}$$
$$n\otimes X_i = X_{n+i \pmod{\frac{M}{2}}}$$
$$X_i\otimes X_j = \left(i+j-1 \pmod{\frac{M}{2}}\right)\oplus\left(i+j-1 \pmod{\frac{M}{2}}+\frac{M}{2}\right)\oplus\left(i+j-1 \pmod{\frac{M}{2}}+\frac{2M}{2}\right).$$

As the cohomology group $H^4(\mathbb{Z}/M\mathbb{Z},\mathbb{C}^\times)$ is trivial, we have from [14, Lemma 4.5] that there exists some associator for both of these quasi-monoidal categories. As $H^3(\mathbb{Z}/M\mathbb{Z},\mathbb{C}^\times) = \mathbb{Z}/M\mathbb{Z}$, there are M associators for each of these quasi-monoidal categories, one for each $\omega\in H^3(\mathbb{Z}/M\mathbb{Z},\mathbb{C}^\times)$. For the choice of cocycle T, we call the M extensions the $\mathbb{Z}/3\mathbb{Z}$ generalised Moore-Read fusion categories of type $(T,M,\omega,+)$ (as there is a unique extension for each $\omega\in H^3(\mathbb{Z}/M\mathbb{Z},\mathbb{C}^\times)$), or $\mathrm{GMR}^T_{\mathbb{Z}/3\mathbb{Z}}(M,\omega,+)$ for short. For the choice of cocycle V, we call the M extensions the $\mathbb{Z}/3\mathbb{Z}$ generalised Moore-Read fusion categories of type $(V,M,\omega,+)$ (again a unique extension for each $\omega\in H^3(\mathbb{Z}/M\mathbb{Z},\mathbb{C}^\times)$), or $\mathrm{GMR}^V_{\mathbb{Z}/3\mathbb{Z}}(M,\omega,+)$ for short.

Case: $1\mapsto_{P\leftrightarrow Q} D_4^{\mathbf{odd}}$

Again we compute the cohomology group $H^2(\mathbb{Z}/M\mathbb{Z},\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z})$ to be $\mathbb{Z}/3\mathbb{Z}$ when 6 divides M, and trivial otherwise. In this case representatives for the non-trivial cocycles are given by

$$T(n,m) = \begin{cases}\mathbf{1}\boxtimes\mathbf{1} \text{ if } n+m<M\\ \mathbf{1}\boxtimes P \text{ if } n+m\geq M,\end{cases}$$

and

$$V(n,m) = \begin{cases}\mathbf{1}\boxtimes\mathbf{1} \text{ if } n+m<M\\ \mathbf{1}\boxtimes Q \text{ if } n+m\geq M,\end{cases}$$

In this case we can twist the multiplication of $\langle (f^{(1)},1)\rangle \subset D_{2N}^- \boxtimes \mathsf{Vec}(\mathbb{Z}/M\mathbb{Z})$ by the cocycles T and V to construct two families of extensions with the same fusion rules as above. Again as $H^3(\mathbb{Z}/M\mathbb{Z},\mathbb{C}^\times) = \mathbb{Z}/M\mathbb{Z}$, there are M such extensions for each choice of cocycle. For the choice of cocycle T, we call the M extensions the $\mathbb{Z}/3\mathbb{Z}$ generalised Moore-Read fusion categories of type $(T,M,\omega,-)$ (as there is a unique extension for each $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$), or $\mathrm{GMR}^T_{\mathbb{Z}/3\mathbb{Z}}(M, \omega, -)$ for short. For the choice of cocycle V, we call the M extensions the $\mathbb{Z}/3\mathbb{Z}$ generalised Moore-Read fusion categories of type $(V, M, \omega, -)$ (again a unique extension for each $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$), or $\mathrm{GMR}^V_{\mathbb{Z}/3\mathbb{Z}}(M, \omega, -)$ for short.

We suspect that an application of [14, Theorem 3.1] would show that the categories $\mathrm{GMR}^T_{\mathbb{Z}/3\mathbb{Z}}(M, \omega_1, +)$ and $\mathrm{GMR}^V_{\mathbb{Z}/3\mathbb{Z}}(M, \omega, +)$ are monoidally equivalent for some choice of $\omega_1, \omega_2 \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$. The same should also hold for the categories $\mathrm{GMR}^T_{\mathbb{Z}/3\mathbb{Z}}(M, \omega_1, -)$ and $\mathrm{GMR}^V_{\mathbb{Z}/3\mathbb{Z}}(M, \omega, -)$. We were unable to prove either of these facts., but this does not matter as our main classification Theorem is not up to monoidal equivalence.

Lemma 4.3.7. *Let C a $\mathbb{Z}/M\mathbb{Z}$ graded extension of* $\mathrm{Ad}(D_4)$. *If C is generated by an object of dimension less than* 2, *then M is even and*

$$C \simeq \langle (f^{(1)}, 1) \rangle \subset D_4^{\pm} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

with $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$, or 6 *divides M and*

$$C \simeq GMR^T_{\mathbb{Z}/3\mathbb{Z}}(M, \omega, \pm) \text{ or } C \simeq GMR^V_{\mathbb{Z}/3\mathbb{Z}}(M, \omega, \pm),$$

with $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$.

Proof. When 6 does not divide N, the proof is exactly the same as the general D_{2N} case.

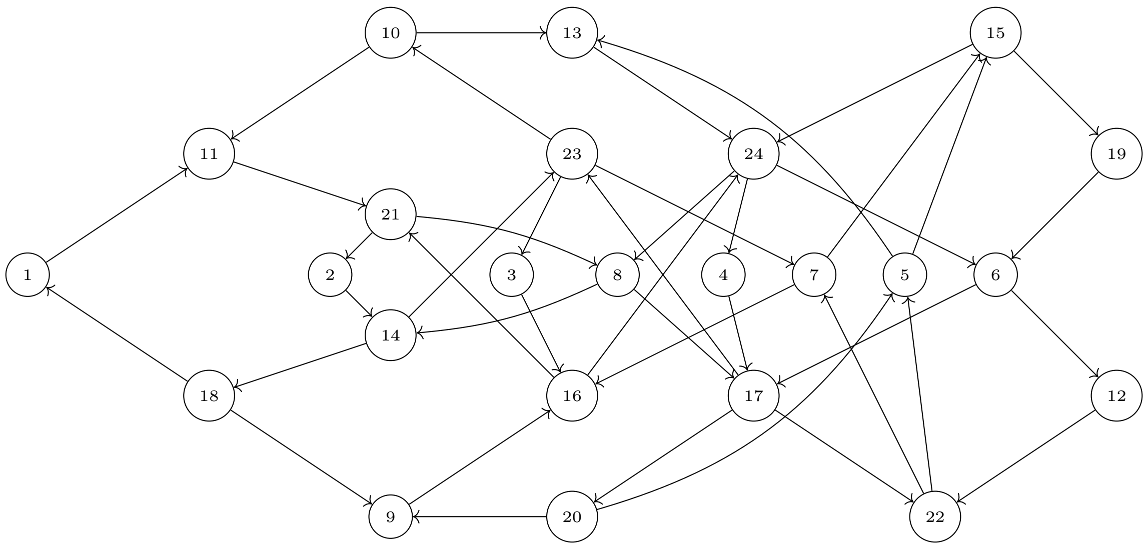
When 6 divides N we can count $6M$ possible $\mathbb{Z}/M\mathbb{Z}$-graded extensions of $\mathrm{Ad}(D_{2N})$, generated by an object of dimension less than 2. The categories in the statement of this Lemma construct all $6M$ of these. □

Cyclic extensions of $\mathrm{Ad}(D_{10})$

Let ϕ be the homomorphism $\mathbb{Z}/6\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(D_{10}))$ determined by $1 \mapsto_{f^{(2)} \mapsto Q \mapsto P} {E_7}^{\text{even}}$. As there are no non-trivial invertible objects in $Z(\mathrm{Ad}(D_{10}))$, we know from extension theory that there are 6 extensions corresponding to this homomorphism, one for each $\psi \in H^3(\mathbb{Z}/6\mathbb{Z}, \mathbb{C}^\times)$. All 6 of these categories have the same fusion rules. We aim to determine these fusion rules.

As $\phi(3) = D_{10}^{\text{odd}}$, we must have that the extension is a $\mathbb{Z}/3\mathbb{Z}$-graded extension of D_{10}. Furthermore we can completely determine the D_{10} bimodules in the $\mathbb{Z}/3\mathbb{Z}$-graded extension. They are E_7 and $\overline{E_7}$. To determine the fusion rules of this $\mathbb{Z}/3\mathbb{Z}$-graded extension of D_{10} we need to determine a D_{10}-balanced bi-functor $E_7 \times E_7 \to \overline{E_7}$. At the level of objects there is a unique such functor, which we avoid writing down for now. This D_{10}-balanced bi-functor gives us the fusion rules for tensoring any two objects in the E_7 component of the grading. The D_{10} bimodule structure of E_7 and $\overline{E_7}$ give us the fusion rules for tensoring an object of D_{10} with any other object in the extension. Counting dimensions of the objects of E_7 and $\overline{E_7}$, we see that every object in the extension must be self-dual. Finally applications of Frobenius reciprocity gives us the full fusion rules. We call these 6 different $\mathbb{Z}/6\mathbb{Z}$-graded extensions of $\mathrm{Ad}(D_{10})$ the $DEE^+(\psi)$ categories (recall there is a unique extension for each $\psi \in H^3(\mathbb{Z}/6\mathbb{Z}, \mathbb{C}^\times)$).

The fusion graph for the generating object of dimension $2\cos\left(\frac{\pi}{18}\right)$ of the $DEE^+(\psi)$ fusion category (which we give the distinguished name Ω) is given by:



The first ten objects of this category are the D_{10}^+ objects, the next seven are the E_7 objects, and the final seven are the $\overline{E_7}$ objects. Numerical approximations of the dimensions of the simple objects in this category are:

$$\{1, 1.96962, 2.87939, 3.70167, 4.41147, 4.98724, 5.41147, 5.67128, 2.87939, 2.87939,$$
$$1.96962, 2.53209, 3.70167, 3.87939, 4.98724, 5.67128, 7.29086,$$
$$1.96962, 2.53209, 3.70167, 3.87939, 4.98724, 5.67128, 7.29086\}$$

These dimensions all live in the field $\mathbb{Q}[\xi_{18}]$. Full fusion rules for the $DEE^+(\psi)$ fusion categories can in Appendix A

Let ϕ be the homomorphism $\mathbb{Z}/6\mathbb{Z} \to \mathrm{BrPic}(\mathrm{Ad}(D_{10}))$ determined by $1 \mapsto_{P\leftrightarrow Q} {E_7}^{\text{even}}$. By the same argument as above we can show the existence of the 6

$DEE^{-}(\psi)$ fusion categories (again one for each $\psi \in H^3(\mathbb{Z}/6\mathbb{Z}, \mathbb{C}^\times)$), which have the same fusion rules as the $DEE^{+}(\psi)$ fusion categories.

Lemma 4.3.1. Let C a $\mathbb{Z}/M\mathbb{Z}$ graded extension of $\mathrm{Ad}(D_{10})$. If C is generated by an object of dimension less than 2, then M is even and

$$C \simeq \langle (f^{(1)}, 1) \rangle \subset D_{10}^{\pm} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

with $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$, or 6 divides M and

$$C \simeq \langle (\Omega, 1) \rangle \subset DEE^{\pm}(\psi) \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z})$$

with $\psi \in H^3(\mathbb{Z}/6\mathbb{Z}, \mathbb{C}^\times)$, and $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)/\mathbb{Z}/6\mathbb{Z}$.

Proof. We begin by classifying homomorphisms from the cyclic group $\mathbb{Z}/M\mathbb{Z}$ to $\mathrm{BrPic}(\mathrm{Ad}(D_{10}))$ that may give rise to extensions generated by an object of dimension less than 2. Consulting the table of Section 4.2 shows that the only bimodules over $\mathrm{Ad}(D_{10})$ that contain an object of dimension less than 2, are D_{10}^{odd}, ${E_7}^{\mathrm{even}}$, and $\overline{E_7}^{\mathrm{even}}$, along with the twistings of each by the 5 non-trivial auto-equivalences of $\mathrm{Ad}(D_{10})$. This leaves us with a total of 18 homomorphisms to consider. Fortunately [14, Theorem 3.1] shows that we only need to homomorphisms $\mathbb{Z}/M\mathbb{Z}$ to $\mathrm{BrPic}(D_{10})$, up to post-composition by the inner automorphisms coming from one of the six bimodules D_{10}^{even}, ${}_{P\leftrightarrow Q}D_{10}^{\mathrm{even}}$, ${}_{P\leftrightarrow f^{(2)}}D_{10}^{\mathrm{even}}$, ${}_{f^{(2)}\leftrightarrow Q}D_{10}^{\mathrm{even}}$, ${}_{f^{(2)}\mapsto P\mapsto Q}D_{10}^{\mathrm{even}}$, and ${}_{f^{(2)}\mapsto Q\mapsto P}D_{10}^{\mathrm{even}}$. As we have described the group structure of $\mathrm{BrPic}(\mathrm{Ad}(D_{10}))$ in Chapter 3, we can directly compute that we only have to consider the 4 homomorphisms:

$$\begin{aligned}
&1 \mapsto D_{10}^{\mathrm{odd}}, \\
&1 \mapsto {}_{P\leftrightarrow Q}D_{10}^{\mathrm{odd}}, \\
&1 \mapsto {}_{f^{(2)}\to Q\to P}{E_7}^{\mathrm{even}}, \\
&1 \mapsto {}_{P\leftrightarrow Q}{E_7}^{\mathrm{even}}.
\end{aligned}$$

We finish our proof in two cases.

Case: M is even and $1 \mapsto D_{10}^{\mathbf{odd}}$ or ${}_{P\leftrightarrow Q}D_{10}^{\mathbf{odd}}$

As $H^2(\mathbb{Z}/M\mathbb{Z}, \mathrm{Inv}(\mathrm{Ad}(D_{10}))) = \{e\}$, and $H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times) = \mathbb{Z}/M\mathbb{Z}$, we have M possible $\mathbb{Z}/M\mathbb{Z}$-graded extensions of $\mathrm{Ad}(D_{10})$ for each of these two homomorphisms. These $2M$ extensions are all realised by the categories

$$\langle (f^{(1)}, 1) \rangle \subset D_{10}^{+} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

and

$$\langle (f^{(1)}, 1)\rangle \subset D_{10}^{-} \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

for $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)$.

Case: 6 divides M and $1 \mapsto_{f^{(2)} \to Q \to P} E_7^{\textbf{even}}$ or $_{P \leftrightarrow Q}E_7^{\textbf{even}}$

Again, as $H^2(\mathbb{Z}/M\mathbb{Z}, \mathrm{Inv}(\mathrm{Ad}(D_{10}))) = \{e\}$ and $H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times) = \mathbb{Z}/M\mathbb{Z}$, we have M possible $\mathbb{Z}/M\mathbb{Z}$-graded extensions of $\mathrm{Ad}(D_{10})$ for each of these two homomorphisms. These $2M$ extensions are all realised by the categories

$$C \simeq \langle (\Omega, 1)\rangle \subset DEE^{+}(\psi) \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z})$$

and

$$C \simeq \langle (\Omega, 1)\rangle \subset DEE^{-}(\psi) \boxtimes \mathsf{Vec}^{\omega}(\mathbb{Z}/M\mathbb{Z}),$$

for $\psi \in H^3(\mathbb{Z}/6\mathbb{Z}, \mathbb{C}^\times)$, and $\omega \in H^3(\mathbb{Z}/M\mathbb{Z}, \mathbb{C}^\times)/(\mathbb{Z}/6\mathbb{Z})$. □

Proof of Theorem 1.0.2

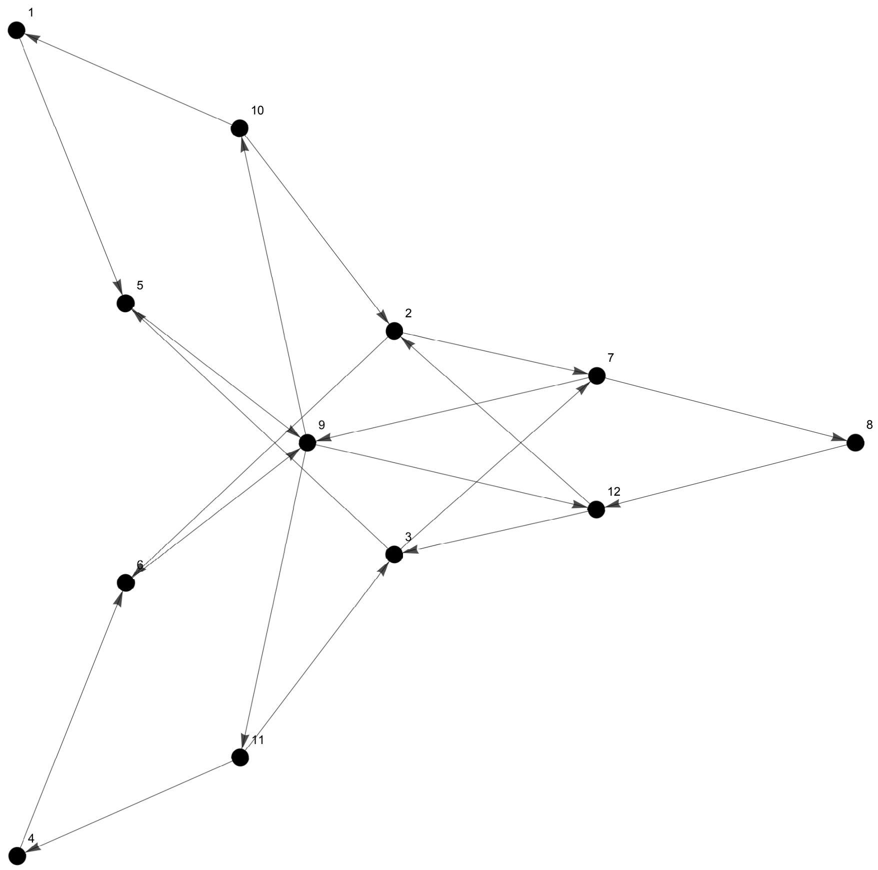
We now put all the pieces together to give the proof of Theorem 1.0.2.

Proof. Let C be a unitary fusion category generated by a normal object X of dimension less than 2. Then by Theorem 1.0.1 C is a unitary cyclic extension of one of $\mathrm{Ad}(A_N)$, $\mathrm{Ad}(D_{2N})$, $\mathrm{Ad}(E_6^{\pm})$, or $\mathrm{Ad}(E_8^{\pm})$. As the categories $\mathrm{Ad}(A_N)$, $\mathrm{Ad}(D_{2N})$, $\mathrm{Ad}(E_6^{\pm})$, and $\mathrm{Ad}(E_8^{\pm})$ are completely unitary, we have that every extension of one of these categories is monoidally equivalent to a unitary category. The previous results of this Subsection classify all cyclic extensions of these categories (except $\mathrm{Ad}(A_7)$) generated by an object of dimension less than 2. In every case, the generating object of dimension less than 2 is normal. □

4.4 The gap at dimension $\sqrt{2+\sqrt{2}}$

As has been previously mentioned, we are not able to give a complete classification of unitary categories generated by a normal object of dimension less than 2. The issue we have is that there are possibly interesting extensions of $\mathrm{Ad}(A_7)$ that we can't construct, or prove non-existence for. Recall that the Brauer-Picard group of $\mathrm{Ad}(A_7)$ is $D_{2\cdot 4}$. The possible interesting extensions of $\mathrm{Ad}(A_7)$, generated by a normal object of dimension less than 2 comes from the copy of $\mathbb{Z}/4\mathbb{Z} \subset D_{2\cdot 4}$. Unfortunately as $H^3(\mathbb{Z}/4\mathbb{Z}, \mathrm{Inv}(\mathrm{Ad}(A_7)))$ is non-trivial, there is no reason to expect that both obstructions can be made to vanish. Furthermore we have found it very difficult to compute the obstruction in $H^3(\mathbb{Z}/4\mathbb{Z}, \mathrm{Inv}(\mathrm{Ad}(A_7)))$ explicitly and check if it vanishes or not. Thus we can not complete the classification at a single dimension, $\sqrt{2+\sqrt{2}}$.

Supposing the possible interesting $\mathbb{Z}/4\mathbb{Z}$-graded extensions of $\mathrm{Ad}(A_7)$ did exist, then we are able to determine the fusion graph for the generating object of dimension $\sqrt{2+\sqrt{2}}$. It is as follows



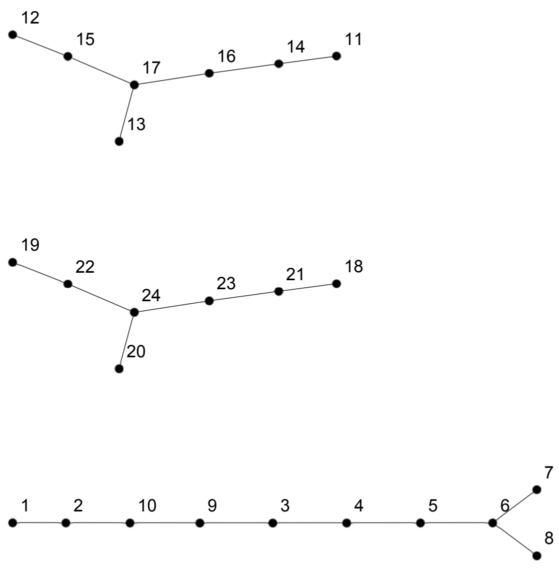
Furthermore we are able to determine associative fusion rules for these possible categories (which provides some evidence for their existence). It would be very interesting to show the existence of these categories, mainly to complete the classification of unitary categories generated by a normal object of dimension less than 2, but also to provide a new example of an interesting new fusion category. We are currently working on a promising line of attack to show the existence of these categories. The idea being that such a category would be a $\mathbb{Z}/2\mathbb{Z}$-graded extension of $(\mathrm{Ad}(A_7) \boxtimes \mathrm{Ising})//\mathsf{Rep}(\mathbb{Z}/2\mathbb{Z})$. The problem now reduces to classifying invertible bimodules over $(\mathrm{Ad}(A_7) \boxtimes \mathrm{Ising})//\mathsf{Rep}(\mathbb{Z}/2\mathbb{Z})$, as the obstruction in $H^3(\mathbb{Z}/4\mathbb{Z}, \mathrm{Inv}(\mathrm{Ad}(A_7) \boxtimes \mathrm{Ising}//\mathsf{Rep}(\mathbb{Z}/2\mathbb{Z})))$ can be seen to vanish at the level of fusion rules. As this line of attack is very recent, we haven't been able to perform the necessary calculations to include the result in this thesis. However we are optimistic that this approach will provide an answer to the existence of the possible interesting $\mathbb{Z}/4\mathbb{Z}$-graded extensions of $\mathrm{Ad}(A_7)$, and thus complete the classification of categories generated by a normal object of dimension less than 2. We will address the full classification in a future publication.

Appendix A

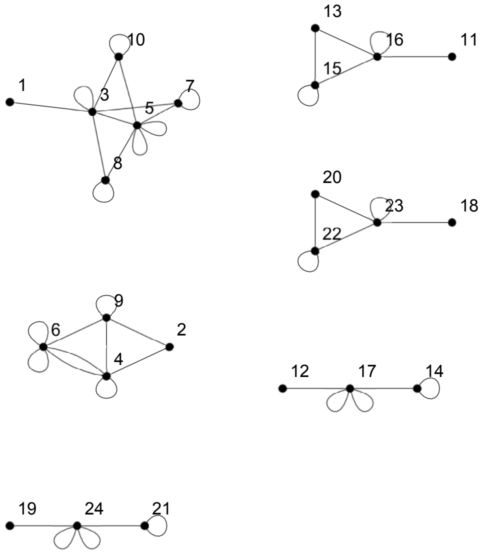
Fusion Rules for the DEE fusion categories

In this appendix we display the fusion graphs Γ_X for each simple object X of the DEE categories. Recall that there are 24 simple objects in this category, and the 11-th object is the tensor generator of dimension $2\cos(\frac{\pi}{18})$.

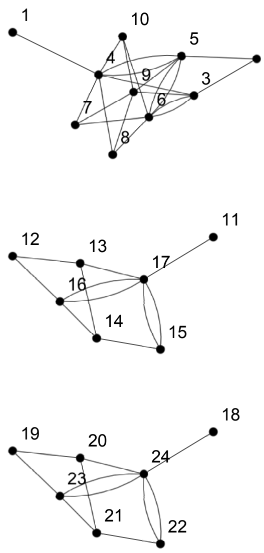
$\Gamma_2 \quad =$



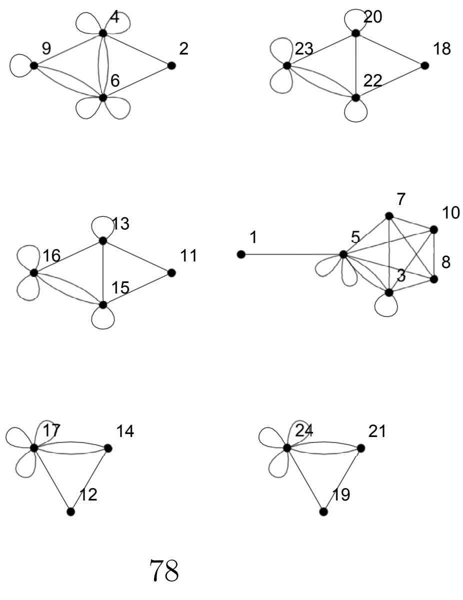
$\Gamma_3 \quad =$



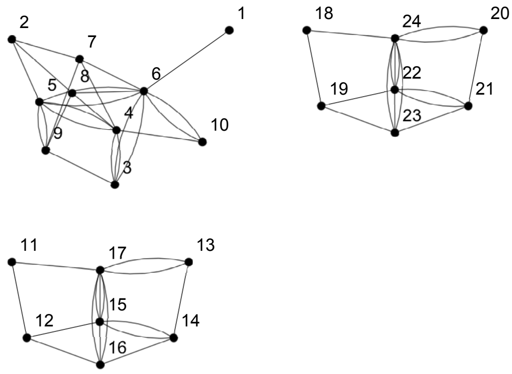
$\Gamma_4 \quad =$



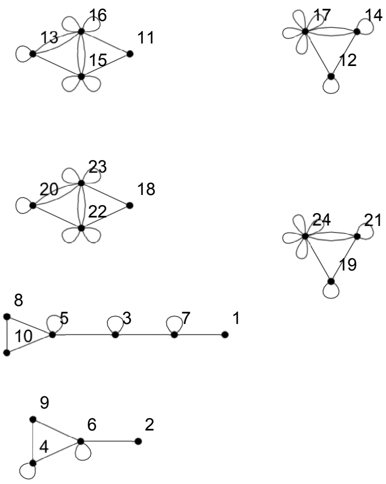
$\Gamma_5 \quad =$



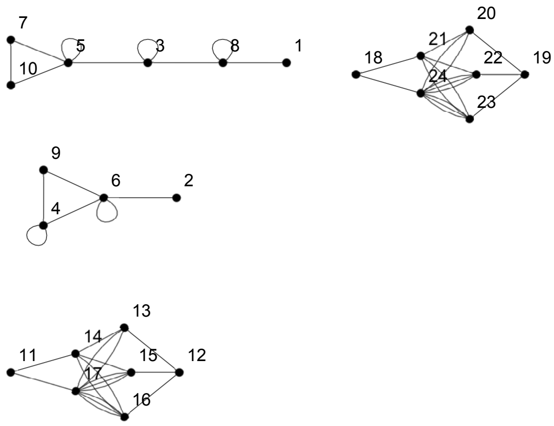
$\Gamma_6 \quad =$



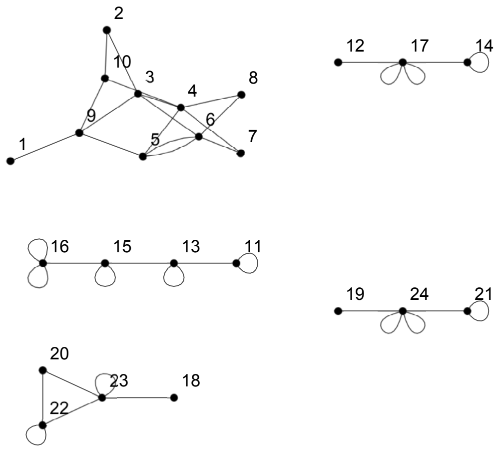
$\Gamma_7 \quad =$



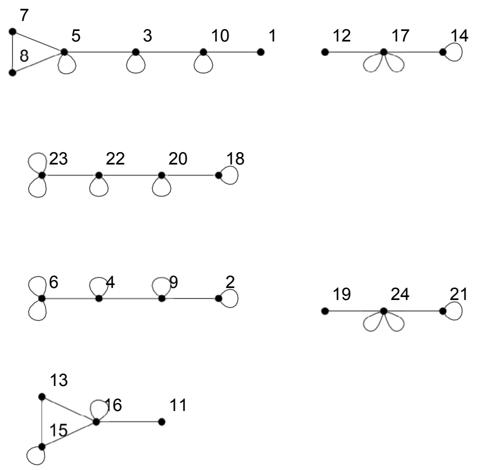
$\Gamma_8 \quad =$



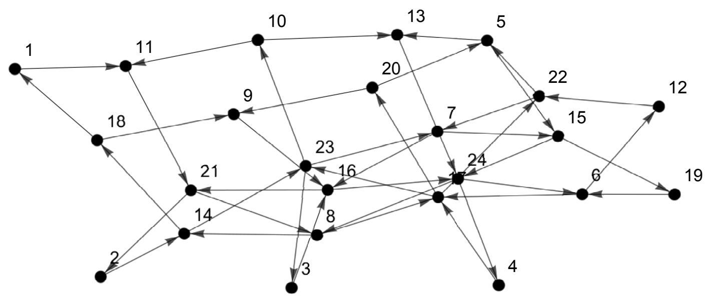
$\Gamma_9 \quad =$



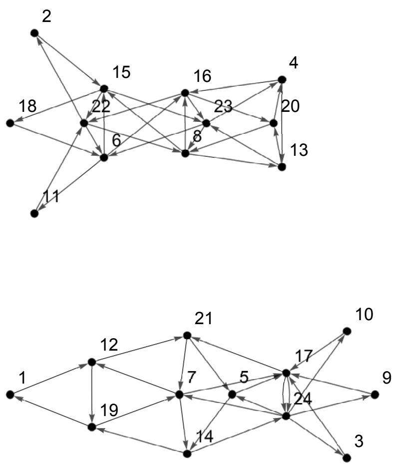
$\Gamma_{10} \quad =$



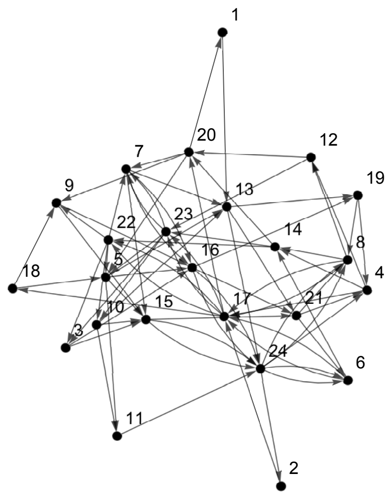
$\Gamma_{11} \quad =$



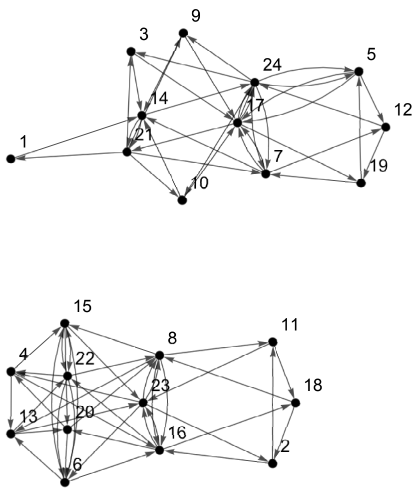
Γ_{12} =



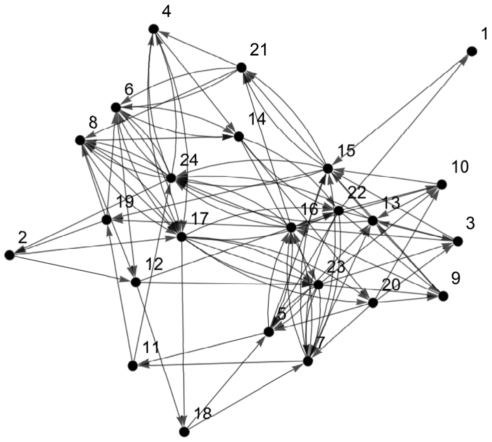
Γ_{13} =



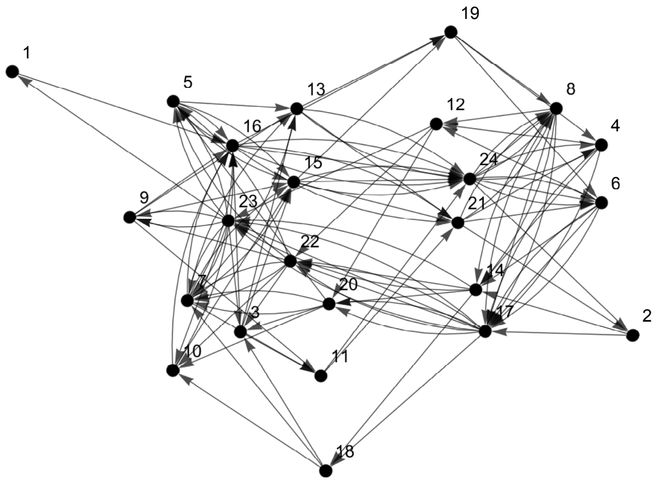
Γ_{14} =



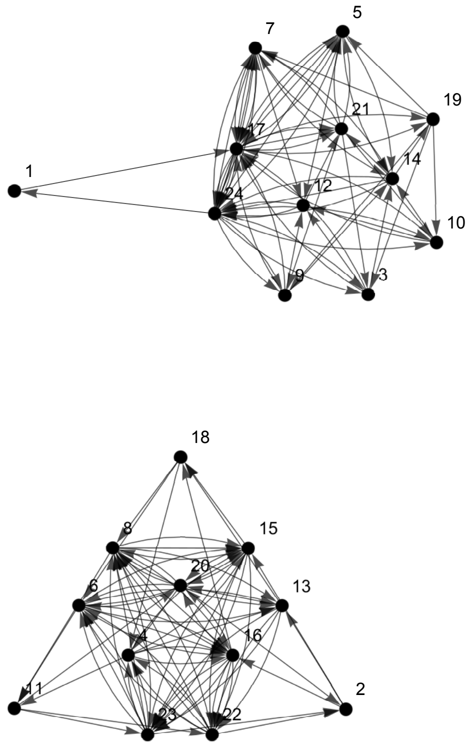
Γ_{15} =



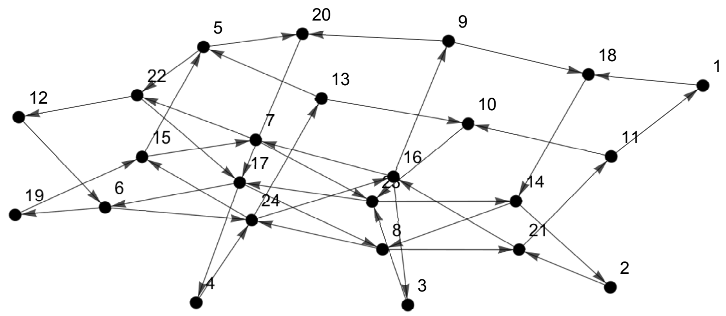
Γ_{16} =



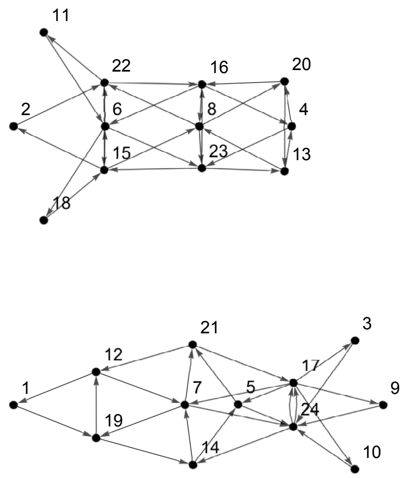
Γ_{17} =



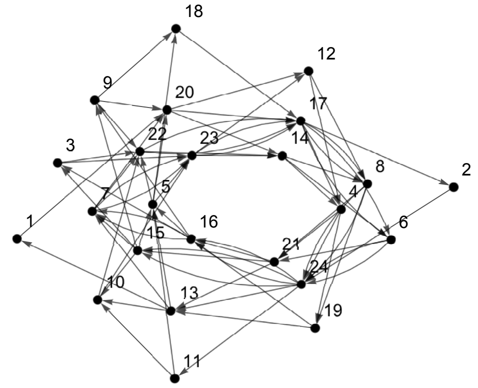
Γ_{18} =



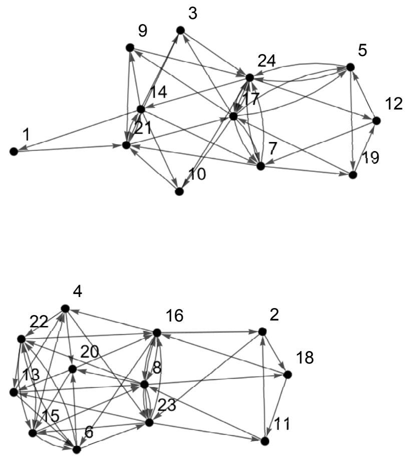
Γ_{19} =



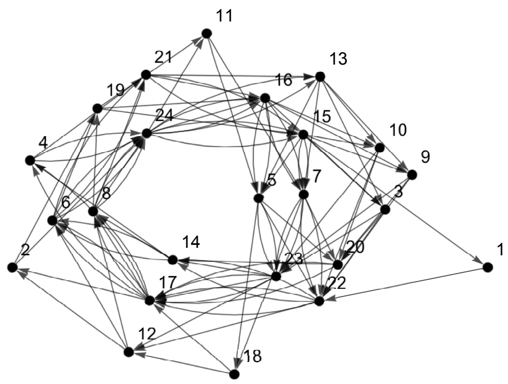
Γ_{20} =



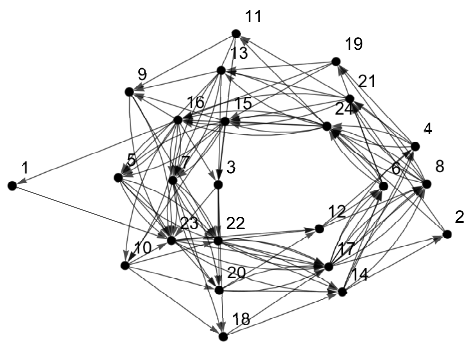
$\Gamma_{21} =$

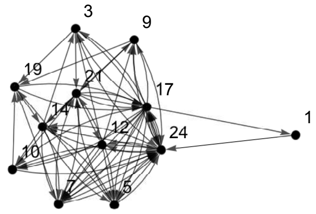


$\Gamma_{22} =$

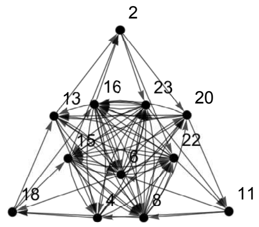


$\Gamma_{23} =$





$\Gamma_{24} \quad =$



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