The Representation Theory of $\mathfrak{sl}_2$

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1 Introduction

$\mathfrak{sl}_2$ is a well understood Lie algebra will multiple applications in special relativity, general relativity and supersymmetry. This paper will outline one of the major results pertaining to $\mathfrak{sl}_2$ in an expository manner. This thesis seeks to completely classify the finite dimensional, complex, irreducible representations of $\mathfrak{sl}_2$ up to isomorphism. To do so the question "What is a Lie algebra?" must be answered and thoroughly studied. From here this paper will develop a toolbox of different techniques that can both help further understand the structure of the Lie algebra as well as build towards the goal of understanding the irreducible representations of $\mathfrak{sl}_2$. This paper will conclude with a thorough proof explicitly classifying all of the irreducible representations of $\mathfrak{sl}_2$. We will do so by defining the action of a representation on each of the three basis elements of $\mathfrak{sl}_2$. From here we will show that this defines a representation and is irreducible. The last step is to show that any other irreducible representation is isomorphic to the representation we defined.

2 Prerequisite Material

Before we can dive deeper into the content of this thesis it is necessary to cover a couple of definitions and results that we will use throughout this paper. These topics can be divided into two categories, general linear algebra knowledge and Lie algebras. We will cover each thoroughly as each will play a fundamental role in understanding $\mathfrak{sl}_2$.

2.1 Linear Algebra

There is one definition we need to cover from linear algebra, this being the definition of a linear map.

Definition 2.1. *Let V and W be vector spaces and $f : V \to W$ a function. We say that f is a linear map if:*

$$f(a+b) = f(a) + f(b)\ \forall a, b \in V$$
$$\alpha f(a) = f(\alpha a)\ \forall a \in V,\ \forall \alpha \in \mathbb{C}.$$

A linear map is a function that preserves both addition and scalar multiplication. These are very useful maps as they allow for separation across addition and for scalars to be brought outside a function. The use of linear maps extend beyond this property though.

Within a finite dimensional space with a basis, a matrix itself is a linear map and every linear map can be represented as a matrix. As this paper pertains to matrices, this result will be utilized frequently. For this reason we will cover an example first.

Example 2.1. *We define* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *with*

$$f = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

We aim to show f *defines a linear map.*

Let $a, b \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. We compute

$$\begin{aligned} f(a+b) &= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}(a+b) \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} a + \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} b \qquad = f(a) + f(b). \end{aligned}$$

Now we check for scalars and compute

$$\begin{aligned} f(\alpha a) &= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \alpha a \\ &= \alpha \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} a \\ &= \alpha f(a). \end{aligned}$$

We conclude f is a linear map.

The next piece we need is a theorem from linear algebra. This theorem relies on the complex numbers being algebraically closed and will be the driving force for our final result.

Theorem 2.1. *Let* V *be a finite dimensional, complex vector space, and* $M : V \to V$ *a linear map. Then there exists* $v \in V - \{0\}$ *and* $\lambda \in \mathbb{C}$ *such that* $M(v) = \lambda v$

This concept is covered in section 5.5 of the textbook *Interactive Linear Algebra*, and is a direct consequence of the Fundemental Theorem of Algebra [Dan19]. Example 2.1 is a linear map without an eigenvalue over a real vector space which emphasizes the importance of utilizing a complex vector space. Alone this result will guarantee the existence of at least one eigenvalue for a matrix with dimension greater than zero. Hold on to these two results for now, we will now dive into Lie theory to set up the final result.

2.2 Lie Algebras

To understand a Lie algebra, one must first understand the fundamental goal of abstract algebra. Within the field of mathematics it is frequent to notice patterns and to try to understand these patterns. Abstract algebra poses the question "can this pattern by 'abstracted' to be applied to different situations?" For example, symmetry was a pattern noticed in the real world and then was abstracted into the concept of groups. Lie algebras are no different. This paper is not going to cover into Lie groups but to understand a lie algebra it must be stated that a lie group is a group (set with composition and all of the axioms) and a differentiable manifold (topological space that is "smooth")[FH91]. At the identity of this manifold emerges the Lie algebra which is the tangent space. The formal definition is stated as:

Definition 2.2. *Let $\mathfrak{g}$ be a vector space with operation:*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}.$$

This operation will be called the Lie bracket or simply bracket and satisfies the following axioms:

$$\begin{aligned} i)\ & f_v(x) := [x, v] \text{ is a linear map } \ \forall\, v \in \mathfrak{g} \\ ii)\ & h_v(x) := [v, x] \text{ is a linear map } \ \forall\, v \in \mathfrak{g} \\ iii)\ & [v, w] = -[w, v] \ \ \forall\, v, w \in \mathfrak{g} \\ iv)\ & [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \ \ \forall\, x, y, z \in \mathfrak{g} \end{aligned}$$

Axiom *iv* is also called the **Jacobi Identity**. There is a lot to unpack with this definition. First and most important are axioms i and ii. These two axioms allow constants and addition within the bracket to be simplified and/or moved outside the bracket. This will prove to be critical when constructing maps from one Lie algebra to the next. From axiom iii we see that the bracket is non-commutative given the bracket is non-zero. Additionally, the elements commuted in the bracket produce the additive inverse of the other which will give rise to the first theorem of the paper.

Theorem 2.2. *Let $\mathfrak{g}$ be a lie algebra and $v \in \mathfrak{g}$, then:*

$$[v, v] = 0$$

Proof.
By Axiom iii:

$$\begin{aligned} [v, v] &= -[v, v] \\ \implies 2[v, v] &= 0 \\ \implies [v, v] &= 0. \end{aligned}$$

□

2.2.1 Endomorphisms of a Vector Space

We have covered the concept of a Lie algebra purely from an abstract point of view. It is for that reason we will cover an example here. The endomorphisms of a vector space is a nice example since the objects in this algebra are linear transformations on the vector space. We will define this Lie algebra as follows:

Definition 2.3. *Let V be a vector space. Then we define:*

$$\text{End}(V) := \{f : V \to V | f \textit{ is linear operator}\}$$

With the bracket $[\cdot, \cdot] := \text{End}(V) \times \text{End}(V) \to \text{End}(V)$ *defined by:*

$$[X, Y] := X \cdot Y - Y \cdot X.$$

A proof that the set of endomorphisms on a vector space is an algebra under the bracket is necessary. First we have the following theorem:

Theorem 2.3. *Let V be a vector space. We have that* $\text{End}(V)$ *is a Lie algebra.*

Proof. First we need show that the bracket operation is a function $\text{End}(V) \times \text{End}(V) \to \text{End}(V)$. To do so let $n, m \in \text{End}(V)$. Therefore

$$n = \begin{bmatrix} a & b \\ c & d \end{bmatrix} m = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \text{ for some } a, b, c, d, e, f, g, h \in \mathbb{C}.$$

We compute

$$\begin{aligned} [n, m] &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} e & f \\ g & h \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} - \begin{bmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{bmatrix} \\ &= \begin{bmatrix} bg - cf & af + bh - be - df \\ ce + dg - ag - ch & cf - bg \end{bmatrix} \\ &\in \text{End}(V) \end{aligned}$$

We want to show that the bracket forms a linear map. Let $t, u, v, w \in \text{End}(V)$ and $a, b \in \mathbb{C}$. Then we compute:

$$\begin{aligned} [a(t+u), b(v+w)] &= a(t+u)b(v+w) - b(v+w)a(t+u) \\ &= ab(t+u)(v+w) - ab(v+w)(t+u) \\ &= ab(tv + tw + uv + uw - vt - vu - wt - wu) \\ &= ab(tv - vt + tw - wt + uv - vu + uw - wu) \\ &= ab([t, v] + [t, w] + [u, v] + [u, w]). \end{aligned}$$

We have now shown that the bracket forms a linear map from both the left and the right. The next claim is that $[v, w] = -[w, v]$.We have that:

$$\begin{aligned}
[v, w] &= v \cdot w - w \cdot v \\
&= -(-(v \cdot w) + w \cdot v) \\
&= -(w \cdot v - v \cdot w) \\
&= -[w, v].
\end{aligned}$$

Let $x, y, z \in \mathrm{End}(V)$. The last claim is that the Jacobi identity holds. We compute:

$$\begin{aligned}
[x, [y, z]] &= [x, yz - zy] \\
&= x(yz - zy) - (yz - zy)x \\
&= xyz - xzy - (yzx - zyx) \\
&= xyz - xzy - yzx + zyx, \qquad (1)
\end{aligned}$$

$$\begin{aligned}
[z, [x, y]] &= [z, xy - yx] \\
&= z(xy - yx) - (xy - yx)z \\
&= zxy - zyx - (xyz - yxz) \\
&= zxy - zyx - xyz + yxz, \qquad (2)
\end{aligned}$$

and

$$\begin{aligned}
[y, [z, x]] &= [y, zx - xz] \\
&= y(zx - xz) - (zx - xz)y \\
&= yzx - yxz - (zxy - xzy) \\
&= yzx - yxz - zxy + xzy. \qquad (3)
\end{aligned}$$

With computations (1)-(3), it follows that:

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

As all of the axioms of a Lie algebra hold for $\mathrm{End}(V)$, it follows that $\mathrm{End}(V)$ is a Lie algebra. □

To see explicitly what this object would look like, consider the case where $V = \mathbb{C}^2$. We will use the standard basis for $\mathrm{End}(\mathbb{C}^2)$,

Example 2.2.

$$\mathrm{End}(\mathbb{C}^2) = \mathrm{span}\{e_1, e_2, e_3, e_4\}$$
$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \; e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \; e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now we must use this basis to compute the bracket table for the basis elements. Since the bracket is a linear operator, this will be sufficient to capture the action of the endomorphisms on $\mathbb{C}^2$. The bracket table is defined by:

$[X, Y]$	e_1	e_2	e_3	e_4
e_1	0	e_2	$-e_3$	0
e_2	$-e_2$	0	$e_1 - e_4$	e_2
e_3	e_3	$e_4 - e_1$	0	$-e_3$
e_4	0	$-e_2$	e_3	0

3 Understanding $\mathfrak{sl}_2$

Before diving into working with algebras and the representations of $\mathfrak{sl}_2$, the algebra $\mathfrak{sl}_2$ itself must be introduced. The set for this algebra is all 2×2 matrices with zero trace. Complex coefficients are used here. This will be important since the complex numbers form an algebraically closed field. The formal definition is as follows:

Definition 3.1. *We define $\mathfrak{sl}_2$ as the complex vector space*

$$\mathfrak{sl}_2 := \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \ \in \mathbb{C} \right\},$$

With the bracket $[\cdot, \cdot] := \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathfrak{sl}_2$ defined by:

$$[X, Y] := X \cdot Y - Y \cdot X.$$

The vector space $\mathfrak{sl}_2$ has the canonical basis $\{H, X, Y\}$, where:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The bracket for $\mathfrak{sl}_2$ on these basis elements is:

$[X, Y]$	H	X	Y
H	0	$2X$	$-2Y$
X	$-2X$	0	H
Y	$2Y$	$-H$	0

As $\mathfrak{sl}_2$ is defined, it is not clear that this is in fact an Lie algebra. For this reason a theorem is necessary.

Theorem 3.1. *We have that $\mathfrak{sl}_2$ is a Lie algebra*

The proof for this theorem is very similar to that of theorem 2.3. It is so similar it would simply involve replacing each instance of $\mathrm{End}(V)$ with $\mathfrak{sl}_2$ since none of the logic was concerned with the vector space itself. For this reason we neglect to include a thorough proof for $\mathfrak{sl}_2$. However we do need to show that the bracket is well defined $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathfrak{sl}_2$. To do so we let

$$n = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} m = \begin{bmatrix} d & e \\ f & -d \end{bmatrix} \text{ for some } a, b, c, d, e, f \in \mathbb{C}.$$

Now we compute

$$\begin{aligned} [n, m] &= \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \begin{bmatrix} d & e \\ f & -d \end{bmatrix} - \begin{bmatrix} d & e \\ f & -d \end{bmatrix} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \\ &= \begin{bmatrix} ad + bf & ae - bd \\ cd - af & ce + ad \end{bmatrix} - \begin{bmatrix} ad + ce & bd - ae \\ af - cd & bf + ad \end{bmatrix} \\ &= \begin{bmatrix} bf - ce & 2(bd - ae) \\ 2(af - cd) & ce - bf \end{bmatrix} \\ &\in \mathfrak{sl}_2. \end{aligned}$$

4 A Tool Box for Lie Algebras

Now that the main object of this paper is established, it is necessary to develop concepts for understanding the behavior of Lie algebras. Each of these objects will build onto one another and once each has been clearly explained it will be clear and precise what a "complete classification of the irreducible representations of $\mathfrak{sl}_2$" entails.

4.1 Lie Algebra Homomorphism

The first such object will be the Lie algebra homomorphism. This is a function that maps one Lie algebra to another. One would expect that this function would preserve the structure of the initial algebra. This is captured in the homomorphism property which is similar to that of rings or groups. Since Lie algebras are vector spaces, this function will also be defined to be a linear map. The definition is as follows:

Definition 4.1. *Let $\mathfrak{g}, \mathfrak{h}$ be Lie Algebras and $\rho : \mathfrak{g} \to \mathfrak{h}$ a function. We say that ρ is a Lie algebra homomorphism if:*

$$\begin{aligned} &1)\rho([x, y]_{\mathfrak{g}}) = [\rho(x), \rho(y)]_{\mathfrak{h}} \ \forall x, y \in \mathfrak{g}. \\ &2a)\rho(ax) = a\rho(x) \ \forall x \in \mathfrak{g} \text{ and } \forall a \in \mathbb{C}. \\ &2b)\rho(x + y) = \rho(x) + \rho(y) \ \forall x, y \in \mathfrak{g}. \end{aligned}$$

Conditions 2a and 2b are equivalent to saying ρ is a linear map.

4.2 Representations of Lie Algebras

Lie algebras are abstract objects which can make certain structures hard to study. This fact can make it challenging to study the behaviors associated with this object. For that reason, it is critical for understanding the structure of $\mathfrak{sl}_2$ that a map is found into a more concrete algebra in which the behavior is well known. Thus, the main idea of representation theory, mapping abstract groups into a more concrete group. A representation is therefore defined as a homomorphism mapping from one Lie algebra into the endomorphisms on a vector space.

Definition 4.2. *Let $\mathfrak{g}$ be a Lie algebra and V a vector space. A representation of $\mathfrak{g}$ is defined:*

$$\rho : \mathfrak{g} \to \mathrm{End}(V)$$

Where ρ is a lie algebra homomorphism.

This definition is mostly helpful but can become confusing when discussing properties and objects that rely on representations. For this reason, note that the vector space V could also be referred to as the representation with the homomorphism implied. This will make more sense when discussing the idea of a sub representation. To be thorough about this concept, consider this example for $\mathfrak{sl}_2$.

Example 4.1. *Let $\rho_1 : \mathfrak{sl}_2 \to \mathrm{End}(\mathbb{C}^3)$ be defined by:*

$$\rho_1(H) =: \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \rho_1(Y) =: \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rho_1(X) =: \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

For brevity we will not discuss a robust proof that ρ_1 is a valid $\mathfrak{sl}_2$ representation. A robust proof for this example would consist of checking the homomorphisms respects three of the nine bracket computations, namely $[X, Y], [X, H]$ and $[Y, H]$. However, the culminating result of this paper will be sufficient to show this is a representation. We say that $\mathbb{C}^3$ is the representation, however it is ρ_1 that captures the characteristics of $\mathfrak{sl}_2$.

4.2.1 Isomorphic Representations

Now that we have discussed representations it is necessary to know when two representations are equivalent. What is being defined here is an isomorphism which is a bijective linear map from one representation to another. This map must respect the actions of the Lie algebra. We say two representations are equivalent or isomorphic if such a function exists. Formally we define:

Definition 4.3. *Let $\mathfrak{g}$ be a Lie algebra and V, W be vector spaces. We define $\rho_1 : \mathfrak{g} \to \mathrm{End}(V)$ and $\rho_2 : \mathfrak{g} \to \mathrm{End}(W)$ to be Lie algebra homomorphisms. An* *isomorphism of representations $V \to W$ is a bijective linear map $f : V \to W$ such that:*

$$\rho_2(x)f(v) = f(\rho_1(x)[v]) \ \forall x \in \mathfrak{g}, \ \forall v \in V.$$

The concept of a isomorphism is best captured with an example. We will begin with constructing two representations. To stay on theme with this paper we will use two $\mathfrak{sl}_2$ representations.

Example 4.2. *Let $\rho_1 : \mathfrak{sl}_2 \to \mathrm{End}(\mathbb{C}^3)$ be defined by:*

$$\rho_1(H) =: \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \ \rho_1(Y) =: \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \ \rho_1(X) =: \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

and let $\rho_2 : \mathfrak{sl}_2 \to \mathrm{End}(\mathbb{C}^3)$ be defined by:

$$\rho_2(H) =: \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \ \rho_2(Y) =: \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \ \rho_2(X) =: \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}.$$

Again for brevity, we will not prove that these are $\mathfrak{sl}_2$ representations. Now it is necessary to define a new function $f : \mathbb{C}^3 \to \mathbb{C}^3$. This new function f must be a bijection and preserve the equality

$$\rho_2(x)f(v) = f(\rho_1(x)[v]) \ \forall x \in \mathfrak{sl}_2, \ \forall v \in \mathbb{C}^3.$$

We must now define a linear map $f : \mathbb{C}^3 \to \mathbb{C}^3$:

$$f = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Proof. We will note that the determinant of f is nonzero. It then follows that f is a bijection. To conclude need to check the homomorphism property on our basis. We now compute across the basis of $\mathfrak{sl}_2$,

$$\begin{aligned} \rho_2(H)f &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\ &= f\rho_1(H). \end{aligned}$$

Next we compute for X,

$$\begin{aligned}
\rho_2(X)f &= \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\
&= f\rho_1(X).
\end{aligned}$$

Lastly we compute for Y,

$$\begin{aligned}
\rho_2(Y)f &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= f\rho_1(Y).
\end{aligned}$$

We can now conclude that our two representations are isomorphic. □

4.2.2 The Adjoint Representations

One example of a representation is the adjoint representation to be referred to as ad. The function is defined as follows:

Definition 4.4. *Let* $\mathfrak{g}$ *be a Lie algebra. We define* $ad : \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$*, where* $ad(x) \in \operatorname{End}(\mathfrak{g})$ *is defined by*

$$ad(x)[y] := [x, y].$$

This definition does not guarantee the adjoint representation is a Lie algebra homomorphism therefore a brief proof illustrating this point is necessary. As defined, if this function defines a homomorphism, it would follow that this is a representation. This is huge as it would guarantee for any Lie Algebra $\mathfrak{g}$ there must exist a representation, namely the adjoint representation. We must prove the following theorem:

Theorem 4.1. *Let* $\mathfrak{g}$ *be a Lie algebra and* V *a vector space. Then* $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}(V)$ *is a Lie algebra homomorphism.*

Proof. There are two parts of this proof. The first is to show ad is a linear map from $\mathfrak{g} \to \operatorname{End}(V)$ and the second is to show the adjoint representation satisfies the homomorphism property. Recall that the Lie bracket is set to be a linear map by the axioms of a Lie algebra. Since the adjoint representation itself is the Lie bracket of two elements it must follow that the adjoint representation is a linear map. Therefore, all that must be shown is that the homomorphism property holds. Since $\mathrm{ad}([X,Y])$ is a function it needs to be considered what the function does to an arbitrary input $Z \in \mathfrak{g}$. We compute

$$\begin{aligned}
\mathrm{ad}([X,Y])[Z] &= [[X,Y],Z] \\
&= -[Z,[X,Y]] && \text{(by axiom iii)} \\
&= [X,[Y,Z]] + [Y,[Z,X]] && \text{(by the Jacobi identity)} \\
&= [X,[Y,Z]] - [Y,[X,Z]] && \text{(by axiom iii)} \\
&= [X,\mathrm{ad}(Y)[Z]] - [Y,\mathrm{ad}(X)[Z]] \\
&= \mathrm{ad}(X)\,\mathrm{ad}(Y)[Z] - \mathrm{ad}(Y)\,\mathrm{ad}(X)[Z] \\
&= [\mathrm{ad}(X),\mathrm{ad}(Y)][Z]
\end{aligned}$$

Therefore it follows that $\mathrm{ad}([X,Y]) = [\mathrm{ad}(X),\mathrm{ad}(Y)]$ and we conclude ad is a lie algebra homomorphism. □

4.3 Subrepresentations

To understand what makes a representation irreducible it is necessary to understand what a subrepresentation is. With other algebraic structures such as a subgroup, what one would expect is that some subset would preserve some property of the structure. For a group, the subgroup preserves composition with the axioms of a group. The goal of a subrepresentation is similar.

Definition 4.5. *Let $\mathfrak{g}$ be a Lie algebra, V a vector space, and $\rho : \mathfrak{g} \to \mathrm{End}(V)$ a representation. We define a subrepresentation of V as a subspace $W \subseteq V$ such that:*

$$\rho(x)[W] \subseteq W \ \forall x \in \mathfrak{g}.$$

It is important to note unlike the representation itself, the subrepresentation is not able to be defined by a function. The sub representation must be defined by the subspace. This is due to the fact that the function ρ is set to define the initial representation and the goal of the subrepresentation is to say that the subspace can act completely independently from the whole algebra and continue to preserve the representation property. This analogous to the concept of a subgroup. Recall from group theory, a subgroup is a subset which preserves the group axioms. This is exactly what is going on here, a subrepresentation is a subspace that preserves the representation property independent of the whole space.

Example 4.3. *Let* $\rho : \mathfrak{sl}_2 \to \mathrm{End}(\mathbb{C}^4)$ *be defined by:*

$$\rho(H) =: \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rho(Y) =: \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rho(X) =: \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

As the goal of this example is to show this is a sub representation, the proof that ρ is a representation will not be illustrated here. We now choose a subspace W. We will check to see if this subspace is invariant of the whole subspace V. We define:

$$W = \left\{ \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix} : a, b \in \mathbb{C} \right\} \subseteq \mathbb{C}^4.$$

Let $w \in W$, then

$$\begin{aligned} \rho(H)w &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -a \\ b \\ 0 \\ 0 \end{bmatrix} \in W. \end{aligned}$$

We also have

$$\begin{aligned} \rho(X)w &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} b \\ 0 \\ 0 \\ 0 \end{bmatrix} \in W. \end{aligned}$$

Finally

$$\begin{aligned} \rho(Y)w &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ a \\ 0 \\ 0 \end{bmatrix} \in W. \end{aligned}$$

It follows that W is a subrepresentation of V.

4.4 Irreducible Representations

The irreducible representations of a Lie algebra lie at the core of this paper. Considering the main goal is to classify explicitly all of the irreducible representations of $\mathfrak{sl}_2$, it is now necessary to state the definition of an irreducible representation.

Definition 4.6. *Let $\mathfrak{g}$ be a Lie algebra, V a vector space and $\rho : \mathfrak{g} \to \mathrm{End}(V)$ a representation. We say V is irreducible if for every subrepresenation $W \subseteq V$, we have that $W = \{0\}$ or $W = V$.*

An irreducible representation is a representation that has no $\mathfrak{g}$-invariant subspaces aside from the trivial cases. This fact will become critical for the methods by which this paper seeks to classify the irreducible representations of $\mathfrak{sl}_2$. We will cover one last example before diving into the main result of the paper. Consider the following,

Example 4.4. *We define $\rho : \mathfrak{sl}_2 \to \mathrm{End}(\mathbb{C}^2)$ as follows:*

$$\rho(H) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ \rho(X) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ \rho(Y) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We aim to show that $\mathbb{C}^2$ is an irreducible representation under ρ.

Proof. Assume there exists some non zero sub representation, namely $W \subseteq \mathbb{C}^2$. Therefore $\rho(H)w \subseteq W$ by definition. By Theorem 2.1 there exists $w \in W - \{0\}$ such that $\rho(H)w = \lambda w$. We will now define a basis for $\mathbb{C}^2$ and for simplicity we will use the standard basis:

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

As v_1, v_2 is a basis for $\mathbb{C}^2$ and $w \in \mathbb{C}^3$ we get

$$w = c_1 v_1 + c_2 v_2 \text{ for some } c_1, c_2 \in \mathbb{C}.$$

Recalling that $\rho(H)w = \lambda w$ we obtain

$$\lambda(c_1 v_1 + c_2 v_2) = \rho(H)(c_1 v_1 + c_2 v_2)$$

It then follows that

$$\begin{aligned} \lambda c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= c_1 \rho(H) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \rho(H) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \implies \lambda c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ \implies \begin{bmatrix} \lambda c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda c_2 \end{bmatrix} &= \begin{bmatrix} c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -c_2 \end{bmatrix}. \end{aligned}$$

We have that $c_1 = 0$ or $c_2 = 0$. To see this assume for contradiction $c_1 \neq 0$ and $c_2 \neq 0$. Then we have

$$\lambda c_1 = c_1 \text{ and } \lambda c_2 = -c_2.$$

We arrive at our contradictition here since λ cannot be both -1 and 1. Hence,

$$w = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \text{ or } w = \begin{bmatrix} 0 \\ c_2 \end{bmatrix}.$$

It then follows we have that either $v_1 \in W$ or $v_2 \in W$.

First assume $v_1 \in W$. Since W is a subrepresentation we have

$$\begin{aligned} \rho(X)v_1 &\in W \\ \implies v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} &\in W. \end{aligned}$$

Now we have the basis for $\mathbb{C}^2$ in W. For case two assume $v_2 \in W$. Since W is a subrepresentation we have

$$\begin{aligned} \rho(Y)v_2 &\in W \\ \implies v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} &\in W. \end{aligned}$$

In both cases we have shown that the standard basis of $\mathbb{C}$ is in W. With the standard basis in W we can now conclude $W = \mathbb{C}^2$. □

5 The Irreducible Representations of $\mathfrak{sl}_2$

The result this paper seeks to demonstrate is a complete classification of all irreducible representations of $\mathfrak{sl}_2$. Recall, a representation is some vector space V with a homomorphism $\rho : \mathfrak{g} \to \text{End}(V)$. We begin by defining a family of irreducible $\mathfrak{sl}_2$ representations.

Definition 5.1. *Let $d \in \mathbb{N}$ and define $V_d := \text{span}_{\mathbb{C}} \{v_0, \cdots, v_{d-1}\}$ and $\rho_d : \mathfrak{sl}_2 \to \text{End}(V_d)$ as the linear extension of:*

$$\begin{aligned} \rho_d(H) &= \delta_{i,j}(d-1-2i) \\ \rho_d(Y) &= \delta_{i-1,j} \\ \rho_d(X) &= \delta_{i+1,j}(i+1)(d-i-1). \end{aligned}$$

Naturally, the first response to this definition would be skepticism that it defines an irreducible $\mathfrak{sl}_2$ representation. It is here that we must acknowledge that this definition was inspired by later results. We address the potential problems with this definition. The function ρ_d is not guaranteed to be a valid Lie algebra homomorphism, based on the definition provided. For this reason an additional theorem will be necessary to confirm the function equips V_d with the structure of an $\mathfrak{sl}_2$ representation. Further we must show that V_d is irreducible. Lastly, there is nothing within the definition that guarantees these are the only representations of $\mathfrak{sl}_2$. Therefore a third theorem and proof will be necessary to show that any $\mathfrak{sl}_2$ representation is isomorphic to the one of V_d defined in definition 5.1.

Theorem 5.1. *Let $d \in \mathbb{N}$, then V_d is an $\mathfrak{sl}_2$ representation.*

Proof. Since the function is defined on the basis vectors, it follows that ρ_d is a linear map. To check the homomorphism property of ρ_d a series of computations must be completed. To begin we verify that $\rho_d([H,Y]) = [\rho_d(H), \rho_d(Y)]$. We compute:

$$\begin{aligned}
[\rho_d(H), \rho_d(Y)] &= [\rho_d(H), \rho_d(Y)]_{i,j} - [\rho_d(H), \rho_d(Y)]_{i,j} \\
&= \sum_{k=0}^{d-1} \rho_d(H)_{i,k}\rho_d(Y)k,j - \rho_d(Y)i,k\rho_d(H)_{k,j} \\
&= (d-1-2i)\delta_{i-1,j} - (d-1-2j)\delta_{i-1,j} \\
&= \delta_{i-1,j}(-2i+2j) \\
&= \delta_{i-1,j}(-2i+2(i-1)) \\
&= -2\delta_{i-1,j} \\
&= -2\rho_d(Y) \\
&= \rho_d([H,Y]).
\end{aligned}$$

Next we must verify $\rho_d([H,X]) = [\rho_d(H), \rho_d(X)]$. We compute:

$$\begin{aligned}
[\rho_d(H), \rho_d(X)] &= [\rho_d(H), \rho_d(X)]_{i,j} - [\rho_d(H), \rho_d(X)]_{i,j} \\
&= \sum_{k=0}^{d-1} \rho_d(H)_{i,k}\rho_d(X)k,j - \rho_d(X)i,k\rho_d(H)_{k,j} \\
&= (d-1-2i)(j)(d-j)\delta_{i+1,j} - (d-1-2j)(i+1)(d-i-1)\delta_{i+1,j} \\
&= (d-2j+1)(j)(d-j)\delta_{i+1,j} - (d-1-2j)(j)(d-j)\delta_{i+1,j} \\
&= 2jd - 2j^2\delta_{i+1,j} \\
&= 2(d-i-1)(i+1)\delta_{i+1,j} \\
&= 2\rho_d(X) \\
&= \rho_d([H,X]).
\end{aligned}$$

Lastly we must verify $\rho_d([X,Y]) = [\rho_d(X), \rho_d(Y)]$. We compute

$$
\begin{aligned}
[\rho_d(X), \rho_d(Y)] &= [\rho_d(X), \rho_d(Y)]_{i,j} - [\rho_d(X), \rho_d(Y)]_{i,j} \\
&= \sum_{k=0}^{d-1} \rho_d(X)_{i,k}\rho_d(Y)k, j - \rho_d(Y)i, k\rho_d(X)_{k,j} \\
&= (i+1)(d-i-1)\delta_{i,j} - j(d-j)\delta_{i,j} \\
&= (id - i^2 - 2i + d - 1)\delta_{i,j} - (jd - j^2)\delta_{i,j} \\
&= (id - i^2 - 2i + d - 1 - id + i^2)\delta_{i,j} \\
&= (d - 1 - 2i)\delta_{i,j} \\
&= \rho_d(H) \\
&= \rho_d([X, Y]).
\end{aligned}
$$

It is sufficient to only show these three computations here. There are six remaining computation. The first three consist of all combinations in which an element of the basis is bracketed with itself. This will hold since the bracket of the same elements will yield the zero vector by Theorem 2.2. The remaining three computations are the commuted versions of the three computations above. Certainly these will hold since negating both sides of the original result will yield the subsequent commuted results. Hence V_d is a representation of $\mathfrak{sl}_2$. □

Now we must introduce our next theorem which claims that this representation is in fact irreducible.

Theorem 5.2. *Let $d \in \mathbb{N}$, then V_d is irreducible.*

What we aim to show with this proof is that for any sub representation W, a special vector v_i can be found. This vector will land back in the subspace W and can be stepped up and down by the actions within the representation until all of the vectors of the whole space are spanned. Therefore it can be concluded $W = V_d$ Therefore all sub representations of V_d are trivial and therefore by definition V_d is irreducible. When the first proof completed, it will be verified that V_d defines an irreducible $\mathfrak{sl}_2$ representation.

Proof. Assume there exists some non-zero subrepresentation, namely $W \subseteq V_d$. Therefore $\rho_d(H)W \subseteq W$ by definition. By Theorem 2.1 there exists $w \in W - \{0\}$ such that $\rho_d(H)w = \lambda w$. As $\{v_i : 0 \leq i < d\}$ is a basis for V_d and $w \in V_d$ we get

$$w = \sum_{i=0}^{d-1} c_i v_i \text{ for some } c_i \in \mathbb{C}.$$

Recall that $\rho_d(H)w = \lambda w$ and we obtain

$$\textstyle\sum_{i=1}^{d-1} \lambda c_i v_i = \sum_{i=0}^{d-1} c_i(d - 1 - 2i) \implies \sum_{i=0}^{d-1} c_i(d - 1 - 2i - \lambda)v_i = 0$$

The v_i's are linearly independent since they are basis vectors. Therefore for all $0 \leq i < d$ we have that $c_i = 0$ or $\lambda = d - 1 - 2i$. If $c_i = 0$ for all $0 \leq i < d$ then $w = 0$ which would contradict $w \in W - \{0\}$. It then follows that $\lambda = d - 1 - 2i$ for some $0 \leq i < d$. Then $c_j = 0$ when $j \neq i$ since $\lambda - d - 1 - 2j = -2(i-j) \neq 0$. Therefore $w = c_i v_i$. As $w \in W$, we get $v_i \in W$.

Now that $v_i \in W$, we claim if $v_j \in W$ then $v_{j-1} \in W$ for all $j \neq 0$. As $v_j \in W$ we get that $v_{j-1} = \rho_d(Y)v_j \in W$ as W is a subrepresentation of V_d. Likewise we claim if $v_j \in W$ then $v_{j+1} \in W$ for all $j \neq d-1$. As $v_j \in W$ we get that $(i+1)(d-i-1)v_{j+1} = \rho_d(X)v_j \in W$ as W is a subrepresentation of V_d. We have now shown that d basis vectors of V_d are in W. It follows that $W = V_d$ □

With this proof completed it is now necessary to show that our definition captures all of the irreducible $\mathfrak{sl}_2$ representations. We will do so with our next theorem.

Theorem 5.3. *Let V be an irreducible representation of $\mathfrak{sl}_2$ with $\dim(V) = d$. Then $V \cong V_d$.*

Let $\rho : \mathfrak{sl}_2 \to \mathrm{End}(V)$ be a Lie algebra homomorphism with V a d-dimensional $\mathbb{C}$ vector space. This proof will be very different from the other proofs in this paper. To present this, a series of lemmas will be presented, each will be distinguished with a distinct claim and the relevance will be summarized after. The culmination will be a clear illustration of the action of $\rho(X), \rho(Y)$ and $\rho(H)$ along with a construction of the isomorphism.

From Theorem 2.1 there exists $v : \rho(H)v = \lambda v$. There may be many such eigenvalues λ for $\rho(H)$. We choose one with $\mathrm{Re}(\lambda)$ maximal among all others.

Lemma 5.4. *We have that $\rho(X)v$ and $\rho(Y)v$ are eigenvectors of $\rho(H)$ with eigenvalues $\lambda + 2$ and $\lambda - 2$ respectively.*

Proof. It is helpful here to note $2\rho(X) + \rho(X)\rho(H) = \rho(H)\rho(X)$. To show this we compute,

$$[\rho(H), \rho(X)] = \rho(H)\rho(X) - \rho(X)\rho(H),$$

and

$$[\rho(H), \rho(X)] = 2\rho(X),$$

therefore

$$\rho(H)\rho(X) - \rho(X)\rho(H) = 2\rho(X) \implies 2\rho(X) + \rho(X)\rho(H) = \rho(H)\rho(X).$$

We then compute:

$$\begin{aligned}
\rho(H)\rho(X)v &= (2\rho(X) + \rho(x)\rho(H))v \\
&= 2\rho(X)v + \rho(X)\rho(H)v \\
&= 2\rho(X)v + \lambda\rho(X)v \\
&= (\lambda + 2)\rho(X)v.
\end{aligned}$$

We then conclude $\rho(X)v$ is an eigenvector of $\rho(H)$. As with the case with $\rho(X)$, note $-2\rho(Y) + \rho(Y)\rho(H) = \rho(H)\rho(Y)$. We then compute:

$$\begin{aligned}\rho(H)\rho(Y)v &= (-2\rho(Y) + \rho(Y)\rho(H))v \\ &= -2\rho(Y)v + \rho(Y)\rho(H)v \\ &= -2\rho(Y)v + \lambda\rho(Y)v \\ &= (\lambda - 2)\rho(Y)v.\end{aligned}$$

We conclude $\rho(Y)v$ is an eigenvector of $\rho(H)$. □

At this point, the general path of the theorem is not clear, however what is known is very explicit. The eigenvalue for $\rho(Y)v$ is $\lambda - 2$ and the eigenvalue for $\rho(X)v$ is $\lambda + 2$. By assumption, $\mathrm{Re}(\lambda)$ is maximal, therefore $\lambda + 2$ cannot be an eigenvalue. We conclude $\rho(X)v = 0$. This will provide a great deal of information which will allow us to find a nice basis for V.

Lemma 5.5. *We have that $\rho(Y)^n v$ is an eigenvector for $\rho(H)$ $\forall n \in \mathbb{N}$ with eigenvalue $\lambda - 2n$.*

Proof. We prove this by induction. Let $N = 1$, then we compute:

$$\rho(Y)^N v = \rho(Y)v.$$

We know $\rho(Y)v$ is an eigenvector for $\rho(H)$ with eigenvalue $\lambda - 2$ from Lemma 5.4, hence the statement holds for $N = 1$. Now assume $\rho(Y)^n v$ is an eigenvector of $\rho(H)$ for some $n \in \mathbb{N}$. Therefore

$$\rho(H)\rho(Y)^n v = \lambda\rho(Y)^n v.$$

We compute:

$$\begin{aligned}\rho(H)\rho(Y)^{n+1}v &= \rho(H)\rho(Y)\rho(Y)^n v \\ &= (-2\rho(Y) + \rho(Y)\rho(H))\rho(Y)^n v \\ &= (-2\rho(Y)\rho(Y)^n v + \rho(Y)\rho(H)\rho(Y)^n v) \\ &= -2\rho(Y)^{n+1}v + \lambda\rho(Y)v\rho(Y)^n v \\ &= -2\rho(Y)^{n+1}v + \lambda\rho(Y)^{n+1}v \\ &= (\lambda - 2n)\rho(Y)^{n+1}v.\end{aligned}$$

We conclude $\rho(Y)^{n+1}v$ is an eigenvector of $\rho(H)$ with eigenvalue $\lambda - 2n$. By induction $\rho(Y)^n v$ is an eigenvector for $\rho(H)$ $\forall n \in \mathbb{N}$. □

This information now grants a set of eigenvectors. In the following proof, this set of eigenvectors will be analyzed again seeking to capture the action of $\rho(X)$, $\rho(Y)$ and $\rho(H)$. But first consider the set

Definition 5.2. $W := \left\{v, \rho(Y)v, \rho(Y)^2 v, \cdots\right\}$

We now have the eigenvalues are $\lambda - 2n$ which means each eigenvalue is distinct. Since the eigenvalues of the elements of W with respect to the operator $\rho(H)$ are distinct, it follows that the elements of W are linearly independent.

Lemma 5.6. *We have $\rho(H)\rho(Y)^n v = (\lambda - 2n)\rho(Y)^n v$ and $\rho(Y)\rho(Y)^n v = \rho(Y)^{n+1}v$*

Proof. For the first claim we compute using Lemma 5.5:

$$\rho(H)\rho(Y)^n v = (\lambda - 2n)\rho(Y)^n v$$

By definition we have $\rho(Y)\rho(Y)^n v = \rho(Y)^{n+1}v$. □

We now need a lemma for the action of $\rho(X)$.

Lemma 5.7. *We have that $\rho(X)\rho(Y)^n v = n(\lambda + 1 - n)\rho(Y)^{n-1}v$.*

Proof. Let $n = 1$ then we compute

$$\begin{aligned}\rho(X)\rho(Y)^n v &= \rho(X)\rho(Y)v \\ &= \rho(H)v + \rho(Y)\rho(X)v \\ &= \lambda v \\ &= 1(\lambda + 1 - 1)\rho(Y)^0 v.\end{aligned}$$

Now assume $\rho(X)\rho(Y)^n v = n(\lambda + 1 - n)\rho(Y)^{n-1}$ for some $n \in \mathbb{N}$ and compute using Lemma 5.5

$$\begin{aligned}\rho(X)\rho(Y)^{n+1}v &= \rho(H)\rho(Y)^n v + \rho(Y)\rho(X)\rho(Y)^n v \\ &= \rho(H)\rho(Y)^n v + n(\lambda + 1 - n)\rho(Y)\rho(Y)^{n-1} \\ &= ((\lambda - 2n) + n(\lambda + 1 - n))\rho(Y)^n v \\ &= (\lambda - 2n + n\lambda + n - n^2)\rho(Y)^n v \\ &= ((n+1)\lambda - n - n^2)\rho(Y)^n v \\ &= (n+1)(\lambda + 1 - (n+1)\rho(Y)^n v.\end{aligned}$$

By induction, it follows that $\rho(X)\rho(Y)^n v = n(\lambda + 1 - n)\rho(Y)^{n-1}v$. □

Not that Lemma 5.6 and Lemma 5.7 give us the action of $\rho(H), \rho(Y)$ and $\rho(X)$ we now want to consider the effect each of these actions has on the elements of W.

Lemma 5.8. *We have $span_{\mathbb{C}}(W) = V$.*

Proof. Note V is an irreducible representation. Therefore, if $span_{\mathbb{C}}(W) \subseteq V$ and is a sub representation, $span_{\mathbb{C}}(W) = V$ or $span_{\mathbb{C}}(W) = 0$. Since $\rho(Y) \in End(V)$, it follows that

$$\rho(Y)^n \in \text{End}(V)\ \forall n \in \mathbb{N}_{\geq 0} \implies \rho(Y)^n v \in V\ \forall n \in \mathbb{N}_{\geq 0}.$$

Therefore $\text{span}_{\mathbb{C}}(W) \subseteq V$. Let $\rho(Y)^n v \in W$ for some $n \in \mathbb{N}_{\geq 0}$. We need show $\rho(X)\rho(Y)^n v \in \text{span}_{\mathbb{C}}(W)$, $\rho(H)\rho(Y)^n v \in \text{span}_{\mathbb{C}}(W)$, and $\rho(Y)\rho(Y)^n v \in \text{span}_{\mathbb{C}}(W)$. We compute

$$\rho(Y)\rho(Y)^n v = \rho(Y)^{n+1} v \in \text{span}_{\mathbb{C}}(W),$$

and

$$\rho(H)\rho(Y)^n v = (d-1-2n)\rho(Y)^n v \in \text{span}_{\mathbb{C}}(W),$$

finally

$$\rho(X)\rho(Y)^n v = n(d-n)\rho(Y)^{n-1} v \in \text{span}_{\mathbb{C}}(W).$$

We then conclude $\text{span}_{\mathbb{C}}(W)$ is a sub representation. Clearly $v \in \text{span}_{\mathbb{C}}(W)$. Therefore $\text{span}_{\mathbb{C}}(W) = V$. □

We have $\rho(Y)^d v = 0$ follows directly from this as V is d dimensional. An interesting fact that follows from the result of the action of $\rho(X)$ is the case involving d.

Lemma 5.9. *We have that $\lambda = d - 1$.*

Proof. We compute usin Lemma 5.7:

$$\rho(X)\rho(Y)^d v = d(\lambda + 1 - d)\rho(Y)^{d-1} = 0$$

Certainly $\rho(Y)^{d-1} \neq 0$ since $span_{\mathbb{C}}(W) = V$. Therefore $d(\lambda + 1 - d) = 0$ and $d \neq 0$. Therefore $\lambda = d - 1$ □

It also follow that the action of $\rho(X)$ is described by $\rho(X)\rho(Y)^n v = n(d - n)\rho(Y)^{n-1} v$ This is helpful as we now fully understand the action of $\rho(H), \rho(X)$, and $\rho(Y)$. The remaining step is to construct the isomorphism to show $V \cong V_d$. Recall $V_d = \text{Span}_{\mathbb{C}} \{v_0, \cdots v_{d-1}\}$ and $V = \text{Span}_{\mathbb{C}} \{v, \rho(Y)v, \cdots, \rho(Y)^{d-1} v\}$. We define a function $f : V_d \to V$ as follows:

Definition 5.3. *We define $f : V_d \to V$:*

$$f(v_i) = \rho(Y)^i v \ \forall v_i \in V_d.$$

It is here that we begin our proof for Theorem 5.3.

Proof. We need to show that this function is in fact a bijection. First to check injectivity. Let $f(v_a) = f(v_b)$ for some $0 \leq a, b < d$.Therefore

$$\begin{aligned} f(v_a) &= f(v_b) \\ \implies \rho(Y)^a v &= \rho(Y)^b v \\ \implies a &= b. \end{aligned}$$

This follows as a direct consequence of $\rho(Y)^a v \in W$ and $\rho(Y)^b v \in W$. Since these are linearly independent basis vectors, the only way for the equation to hold is for the number of applications of $\rho(Y)$ to be equal. Therefore we have $v_a = v_b$. Now we must check that f is surjective. Let $y \in V$. Therefore $y = \rho(Y)^a v$ for some $0 \leq a < d$. We note $v_a \in V_d$ and $f(v_a) = \rho(Y)^a v$, so we can conclude that f is surjective. It then follows that f is a bijection. Lastly we need to check the isomorphism property across the basis of $\mathfrak{sl}_2$. Let $v_a \in V_d$ for some $0 \leq a < d$. First we compute on X:

$$\begin{aligned}\rho(X)f(v_a) &= \rho(X)\rho(Y)^a v\\ &= a(d-a)\rho(Y)^{n-1}v\\ &= f((i+1)(d-(i+1))\delta_{i+1,j}v_a)\\ &= f(\rho_d(X)v_a).\end{aligned}$$

Next we compute on Y:

$$\begin{aligned}\rho(Y)f(v_a) &= \rho(Y)\rho(Y)^a v\\ &= \rho(Y)^{n+1}v\\ &= f(\delta_{i-1,j}v_a)\\ &= f(\rho_d(Y)v_a).\end{aligned}$$

Last we compute on H:

$$\begin{aligned}\rho(H)f(v_a) &= \rho(H)\rho(Y)^a v\\ &= (d-1-2a)\rho(Y)^n v\\ &= f((d-1-2i)\delta_{i,j}v_a)\\ &= f(\rho_d(H)v_a)\end{aligned}$$

Thus $\rho : V_d \to V$ is an isomorphism and it follows that $V \cong V_d$. □

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