THE IDEAL STRUCTURE OF A STEINBERG ALGEBRA

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1. Introduction

Leavitt path algebras were first introduced by Abrams and Aranda Pino in [4] as an algebraic analogue of graph C^*-algebras. This analogue is in the sense that the C^*-algebra of a graph is the norm completion of its Leavitt path algebra [21, Theorem 7.3]. These algebras were initially defined only for row-finite directed graphs with no sinks, but the later work [3] generalizes these algebras to any directed graph. Many of the theorems initially proven for the row-finite, no sinks case have been shown to hold true in the arbitrary case as well under more general conditions, for an example see [21, Theorem 6.18]. Leavitt path algebras form a broad class of algebras that cover many well known classes of algebras such as Matrix algebras and Laurent polynomial algebras [4, Example 1.4]. It turns out that many of the properties of a Leavitt path algebra are uniquely determined by the structure of the underlying graph. For examples of this see [4, Theorem 3.11] for a characterization of simple Leavitt path algebras, and [2, Theorem 4.2] for the description of the center of simple Leavitt path algebras. Because of this correlation between the structure of the graph and its Leavitt path algebra we can construct highly non-trivial examples of algebras satisfying certain conditions by simply constructing a graph that satisfies the equivalent conditions.

A generalization of Leavitt path algebras of row-finite graphs with no sinks was introduced by Aranda Pino, (J.) Clark, an Huef and Raeburn in [7] in which they construct a class of algebras associated to row-finite higher rank graphs (introduced in [12]) with no sinks. They call this new class of algebras Kumjian-Pask algebras. Since their introduction in 2011 there have been several papers published on the structure of these algebras. These include a classification of the ideal structure [7], as well as a classification of the center [1]. As in the case with Leavitt path algebras the structure of Kumjian-Pask algebras seems to be uniquely determined by the structure of the underlying k-graph. This class of algebras include several interesting algebras not covered under Leavitt path algebras. For example a Laurent polynomial algebra in multiple dimensions can be realized as a Kumjian-Pask algebra [7, Remark 7.2]

Let G be an ample groupoid and R a commutative ring with identity. The Steinberg algebra $A_R(G)$ was first introduced in [18] as a model for inverse semi-group algebras. They were also examined in [9] with $R := \mathbb{C}$ as a generalization of Kumjian-Pask algebras, and in particular, Leavitt path algebras of row-finite directed graphs with no sinks. These algebras are of use to study as they seem to include examples of algebras that are not Kumjian-Pask algebras (see [9, Example 5.9]). They also allow the use of the groupoid structure in proofs which may provide new insight on the theory of Kumjian-Pask algebras as well as simplify arguments in existing proofs.

Steinberg algebras have not yet been extensively studied in the literature. Since their introduction in [18] there have been 5 papers published [19][11][8][9][6] on the subject. These papers have mostly dealt with generalizing results known for Kumjian-Pask algebras and the appropriate Leavitt path algebras up to the more general case of Steinberg algebras.

Steinberg, along with introducing them, proves several theorems about Steinberg algebras in [18]. He characterizes the center of $A_R(G)$ in the general case as well giving necessary and sufficient conditions for the algebra to be unital. He also shows that the class of algebras known as Semigroup algebras are in fact Steinberg algebras. This provides another source of examples that may prove useful in further understanding these algebras. In his follow up paper [19], Steinberg studies the simplicity, primitivity and semiprimitivity of Steinberg algebras.

In [9] the authors give a proof that the class of Steinberg algebras generalizes the class of Kumjian-Pask algebras. They also generalize several known results about Kumjian-Pask algebras up to the Steinberg algebra setting. An important one of these is a Cuntz-Krieger uniqueness theorem which characterizes the injectivity of a homomorphism from a Steinberg algebra to an R-algebra when the underlying groupoid is *topologically principle*. Cuntz-Krieger uniqueness theorems were initially introduced for graph $C*$-algebras and proved to be an important tool for classifying ideals in these algebras. For example in [17] the author uses the uniqueness theorem to completely classify the graded ideal structure. Because of the usefulness of the Cuntz-Krieger uniqueness theorem analogous algebra versions of have been established for Leavitt path algebras [21][20] and Kumjian-Pask alebras [7].

Still in the case where $R := \mathbb{C}$ the paper [6] makes the first attempt at classifying the ideal structure of these algebras by finding a necessary and sufficient condition for a Steinberg algebra to be simple. They also characterize the ideal structure of a groupoid C^*-algebra when the underlying groupoid is *strongly effective*(see Section 2) and second countable. This later result is of particular importance to us as we base the main proof of this thesis on the methods of [6, Corollary 5.9].

The authors of [11] give a sufficient condition for two Steinberg algebras to be Morita equivalent (see [15, Definition 1]). They show that if groupoids are equivalent in the sense of [11, Definition 4.1] they have Morita equivalent Steinberg algebras. This result is analogous to a similar theorem known for groupoid C^*-algebras.

In [8] we attempt to generalize the results of [6] and [9] to the case where R is any commutative ring with identity. This was motivated by a similar generalization of Leavitt path algebras as done in [21] by Tomforde. Here the we combine the techniques used in

[21] to adjust the proofs of several theorems in [6] and [9] to prove these broader results. In particular we use the idea of a *basic ideal* from [21] which stops the ideal structure of the underlying ring effecting the ideal structure of the algebra, allowing us to focus on the effect of the groupoid structure. In this thesis we are considering Steinberg algebras over arbitrary commutative rings with identity so we will also use the notion of a basic ideal. Of the theorems proved in this paper of particular relevance to this thesis is an extended version of a Cuntz-Krieger uniqueness theorem that applies to Steinberg algebras over over arbitrary commutative rings with identity. We also relax the condition of the groupoid being topologically principle to the weaker condition of effective. In addition we give the complete classification of basically simple Steinberg algebras.

In this thesis we attempt to further classify the ideal structure of Steinberg algebras. In section 4 we prove an analogous result to a known theorem for groupoid C^*-algebras shown in [6, Theorem 4.9]. This theorem characterizes the basic ideal structure of $A_R(G)$ when G satisfies a condition which we call *strongly effective*. We also weaken the hypothesis that was used in the groupoid C^*-algebra version as we don't require G to be second countable. Due to the similar nature of the proofs it may be possible to prove this stronger result for groupoid C^*-algebras using similar techniques that we use in our proof. C^*-algebraists may find the result of section 4 interesting as it shows that the norm of a groupoid C^*-algebra does not effect the ideal structure in this setting.

In section 4 we use the basic ideal theory we have already established in section 4 to study the more general ideal theory. As we are now dealing with non-basic ideals we now also have to consider the structure of the ring as well as the structure of the groupoid as both can effect the ideal structure of a Steinberg algebra as we show in our preliminaries. Non-basic ideals were studied in the Kumjian-Pask algebra setting where the authors create a map from the set of ideals in the ring to the ideals of a Kumjian-Pask algebra. In particular they show that this map is bijective when the underling higher rank graph of the algebra satisfies the equivalent conditions of effective and minimal. Motivated by this idea we extend the map they introduce so that it also takes into account the structure of the underlying groupoid, and hence includes more of ideals of the algebra. We initially hoped to show this map was a bijection when G is strongly effective but unfortunately we have not been able to prove it is so. We show that the map is injective in general and bijective when G is minimal and effective which is the extention of the Kumjian-Pask alegbra result. We leave that the map is surjective when G is strongly effective as a conjecture that could be investigated as an extension to this thesis. We include a lemma at the end of this thesis that summarizes our work on the conjecture.

2. Preliminaries

2.1. **Groupoids.** A groupoid is a generalization of a group where composition is only defined between certain pairs of elements. The easiest way to define a groupoid is as a small category with inverses. However in this thesis we give a more intuitive definition for the reader that is equivalent. Following [16, Definition 2.1] we define a groupoid as follows

Definition 2.1. Let G be a set and let $G^{(2)}$ be a subset of $G \times G$. Suppose there is a map $(\alpha, \beta) \to \alpha\beta$ from $G^{(2)} \to G$ and an involution $\alpha \to \alpha^{-1}$ on G, then we call G a groupoid if the following conditions hold:
(1) If (α, β) and (β, γ) are in $G^{(2)}$, then so are $(\alpha\beta, \gamma)$ and $(\alpha, \beta\gamma)$, and the equation, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds.
(2) For all $\alpha \in G$, $(\alpha^{-1}, \alpha) \in G^{(2)}$ and if $(\alpha, \beta) \in G^{(2)}$, then $\alpha^{-1}(\alpha\beta) = \beta$ and $(\alpha\beta)\beta^{-1} = \alpha$.

We define the functions r and s on G by the formulae $r(\alpha) = \alpha\alpha^{-1}$ and $s(\alpha) = \alpha^{-1}\alpha$, these are called the range and source maps respectively. These maps give us a convenient way to see if elements are composable in a groupoid with the following remark

Remark 2.2. For any $\alpha, \beta \in G$ we have $(\alpha, \beta) \in G^{(2)}$ if and only if $s(\alpha) = r(\gamma)$.

Suppose $(\alpha, \beta) \in G^{(2)}$, then $\alpha\beta = \gamma$ for some $\gamma \in G$ and so $\alpha = \gamma\beta^{-1}$. Thus

$$s(\alpha) = s(\gamma\beta^{-1}) = s(\beta^{-1}) = r(\beta).$$

Suppose $s(\alpha) = r(\gamma)$, then $\alpha^{-1}\alpha = \beta\beta^{-1}$. We can left multiply this equation by α and right multiply by β by part (2) of the definition of a groupoid, this leaves us with $\alpha\beta = \alpha\beta$ which implies that $(\alpha, \beta) \in G^{(2)}$.

We call the common image of r and s the unit space of G and denote it $G^{(0)}$.

We define the composition between two subsets A and B of a groupoid as

$$AB := \{\alpha\beta : \alpha \in A, \beta \in B, s(\alpha) = r(\beta)\}.$$

In the special case where A and B are both subsets of units this composition reduces to set theoretic intersection. This follows as the range and source of a unit μ are both equal to μ, so we get

$$\begin{aligned} AB &= \{\mu\nu : \mu \in A, \nu \in B, s(\mu) = r(\nu)\} \\ &= \{\mu\nu : \mu \in A, \nu \in B, \mu = \nu\} \\ &= \{\mu\mu : \mu \in A, \mu \in B\} \\ &= \{\mu : \mu \in A, \mu \in B\} \end{aligned}$$

as units are idempotents by part (2) of the definition of a groupoid . To simplify our notation, for a subset $D \subseteq G^{(0)}$ we define

$$G_D := s^{-1}(D) \text{ and } G^D := r^{-1}(D).$$

We now present a few examples of groupoids.

Examples 2.3. (1) An easy example of a groupoid is a group. Here $G^{(2)} = G \times G$ and the range and source maps map to the identity e of the group and hence $G^{(0)} = \{e\}$.

(2) The set of all invertible square matrices is a groupoid under multiplication with composition defined between matrices of the same dimensions. Here the range and source maps are equal and map an $n \times n$-matrix to the n-dimensional identity matrix.

(3) Given a countable directed row-finite graph with no sources E we can construct a corresponding groupoid first introduced in [14]

$$G_E := \{(x, n, y) : x, y \in E^\infty, \sigma^l(x) = \sigma^k(y), l - k = n\}$$

where E^∞ is the set of all infinite paths in the graph, and the function σ^n removes the first n edges off a path. With composition defined by

$$(x, n, y)(y, m, z) := (x, n + m, z)$$

and inverse by

$$(x, n, y)^{-1} := (y, -n, x),$$

it is shown in [14] that G_E is a groupoid.

We call G a topological groupoid if G is endowed with a topology such that the inverse and composition maps are continuous. Note that this implies that the range and source maps are continuous as well as they are the groupoid composition of the identity and inverse map.

A bisection of G is a subset B of G such that both $r|_B$ and $s|_B$ are homeomorphisms. The set multiplication of bisections of a groupoid is nicely behaved with the product of two bisections itself being a bisection [18, Proposition 2.3].

We call G étale if both r and s are local homeomorphisms and ample if it is étale and the topology of G has a basis of compact bisections. For the remainder of this thesis we will only be working exclusively with Hausdorff ample groupoids. The reason we do this becomes clear once we introduce Steinberg Algebras.

Hausdorff ample groupoids have several nice properties that we breifly state now. An immediate corollary to G being ample is that G is locally compact, this is because we can find a compact open bisection around every point and hence a compact set. As ample implies étale [9, Lemma 2.1] we have that the functions r and s are local homeomorphisms, and hence open maps. In particular the unit space $G^{(0)}$ is open as it is the image of G under the maps r and s. It can also be shown as in [16, Section 2.1] that when G is Hausdorff, $G^{(0)}$ is closed. Thus for the rest of this thesis we can assume that the unit space of whatever groupoid we are working with is both open and closed (clopen).

A groupoid G is said to be *effective* if every open bisection $B \subseteq G \setminus G^{(0)}$ contains an element α such that $r(\alpha) \neq s(\alpha)$. This definition is given in [6, Lemma 3.1] along with several other equivalent characterizations. One of these equivalent characterizations that is particularly useful in the classification of ideals in $A_R(G)$ is that a groupoid is effective if and only if for every compact $B \subseteq G \setminus G^{(0)}$ and open $K \subseteq G^{(0)}$ there exists a non-empty open $K_0 \subseteq K$ such that $K_0 B K_0 = \emptyset$. We expand on this characterization below with a lemma that we use in section 5.

Lemma 2.4. *Let G be an Hausdorff ample groupoid. Then G is effective if and only if for every open $K \subseteq G^{(0)}$ and compact $B \subseteq G$ such that K and B are disjoint, there exists nonempty compact open $K_0 \subseteq K$ such that $K_0 B K_0 = \emptyset$.*

Proof. Suppose G is effective and fix B and K as in the statement of the proof. We write $B_{G^{(0)}} = B \cap G^{(0)}$ and $B_G = B \setminus G^{(0)}$, these sets are compact as $G^{(0)}$ is clopen. We use the equivalent definition of effective [6, Lemma 3.1 (1) $\rightarrow$ (4)] to get nonempty open $K_0 \subseteq K$ such that $K_0 B_G K_0 = \emptyset$. We can assume K_0 is compact as G is ample, so if K_0 was not compact we could select a compact open subset of K_0 that still satisfies the required condition. We can see that K_0 is disjoint from $B_{G^{(0)}}$ as it is a subset of K, thus as composition of sets of units in a groupoid is the set theoretic intersection we get $K_0 B_{G^{(0)}} K_0 = \emptyset$. Therefore $K_0 B K_0 = K_0(B_G \cup B_{G^{(0)}})K_0 = K_0 B_G K_0 \cup K_0 B_{G^{(0)}} K_0 = \emptyset$.

Now suppose the reverse assumption and let $B \subseteq G \setminus G^{(0)}$ be compact and $K \subseteq s(B)$ be open. These sets are clearly disjoint so we get non-empty compact open $K_0 \subseteq K$ such that $K_0 B K_0 = \emptyset$. As $K_0 \subseteq s(B)$ there exists $\gamma \in B$ with $s(\gamma) \in K_0$ but we have that $K_0\{\gamma\}K_0 = \emptyset$ so $r(\gamma) \notin K_0$ and hence $r(\gamma) \neq s(\gamma)$. ☐

For groupoids there is a similar definition called *topologically principle* (see the preliminarys of [9]). This is a stronger condition than effective in general [5, Remark 3.7], however when G is second countable these definitions coincide [5, Remark 3.6]. As stated in the introduction several results shown to hold for topologically principle groupoids have also been shown to be true for effective groupoids.

We call a subset D of the unit space *invariant* if $s^{-1}(D) = r^{-1}(D)$, or equivalently $r(\gamma) \in D$ if and only if $s(\gamma) \in D$ for all $\gamma \in G$. If D is invariant then G_D is itself a groupoid with unitspace D and $G_D = G^D$.

Remark 2.5. It suffices to show when proving something is invariant to show that $r(\gamma) \in D$ implies $s(\gamma) \in D$. If this is the case then suppose $s(\gamma) \in D$, then $r(\gamma^{-1}) \in D$ and so $r(\gamma) = s(\gamma^{-1}) \in D$.

Remark 2.6. For any subset K of the unit space, we can take $[K] := r(s^{-1}(K))$ to get the *saturation* of K, that is the smallest invariant set containing K. To show this set is invariant let $r(\gamma) \in r(s^{-1}(K))$, then $r(\gamma) = r(\alpha)$ for some $\alpha \in G^{(0)}$ with $s(\alpha) \in K$. This implies that

$$s(\gamma) = r(\gamma^{-1}) = r(\gamma^{-1}\alpha) \in s^{-1}(K),$$

as $s(\gamma^{-1}\alpha) = s(\alpha) \in K$.

To see this is the smallest such set let $D \subseteq G^{(0)}$ be invariant such that $K \subseteq D \subseteq [K]$. Let $\mu \in [K]$, then $\mu = r(\gamma)$ for some $\gamma \in G$ with $s(\gamma) \in K$. This implies that $s(\gamma) \in D$, thus as D is invariant, $\mu = r(\gamma) \in D$.

We say that a groupoid G is *minimal* if $G^{(0)}$ has no nontrivial open invariant subsets.

We say a groupoid is *strongly effective* if for every closed invariant set $D \subseteq G^{(0)}$ we have that G_D is effective. Note that this definition has not been used in the literature before. We choose to give this property a name as it will appear heavily in this thesis. A similar version of this property has been previously used in the literature. The example mainly of interest to us is in [6, Theorem 5.9] where they use the property that G_D is topologically principle for each closed invariant $D \subseteq G^{(0)}$.

We now present a few examples of Hausdorff ample groupoids.

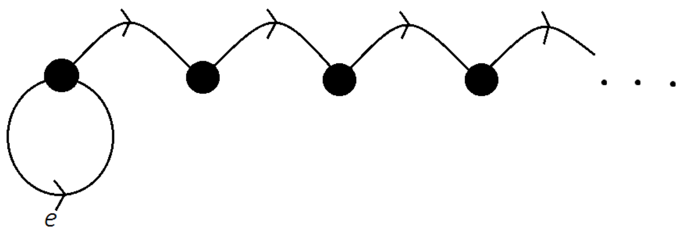
Examples 2.7. (1) Let E be a directed row-finite graph with no sinks, and consider the groupoid G_E generated by this graph as in 2.3. We equip this groupoid with the topology described in [14, section 2]. For any finite path α in E we define the set

$$Z(\alpha) := \{x \in P(E) : x_1 = \alpha_1, \dots, x_{|\alpha|} = \alpha_{|\alpha|}\},$$

where $P(E)$ is the set of all finite paths in E. Then the sets

$$Z(\alpha, \beta) := \{(x, k, z) : x \in Z(\alpha), y \in Z(\beta), k = |\beta| - |\alpha|, x_i = y_{i+k} \text{ for } i > |\alpha|\}$$

form a basis of compact open bisections for G_E[14, Proposition 2.6], hence G_E is Hausdorff ample. In particular every cycle in E has an exit then G_E is effective[13, Lemma 3.4]. While it may appear to the reader the the definitions of strongly effective and effective are equivalent this is not the case. The following graph demonstrates this as every loop has an exit, however the subset $S = \{eee\ldots\}$ is closed and invariant. When we restrict our groupoid to this subset we can see it is isomorphic to $\mathbb{Z}$ and hence not effective.



(2) Let G be a finite group with the discrete topology, then G is a minimal Hausdorff ample groupoid. However G is not effective except for the trivial group as the range and source maps in a group are always equivalent.

2.2. **Steinberg Algebras.** Let R be a commutative ring with identity, for any function $f : G \to R$ we define the *support* of f to be

$$\mathrm{supp}(f) := \overline{\{\gamma \in G : f(\gamma) \neq 0\}}.$$

We define $A_R(G)$ as the set of all locally constant functions $f : G \to R$ with compact support. In this case locally constant means that at every point of G there exists an open neighborhood of that point on which f is constant.

We define addition and scalar multiplication on this set to be pointwise, and multiplication between elements $f, g \in A_R(G)$ as follows

$$(f * g)(\alpha) = \sum_{r(\beta)=r(\alpha)} f(\beta)g(\beta^{-1}\alpha) \text{ for every } \alpha \in G.$$

We can see from Lemma 2.2 that this sum is over all β such that the product $\beta^{-1}\alpha$ is defined. It is proven in [18, Proposition 3.5] that with these operations the set $A_R(G)$ becomes an R-algebra. It is clear now why we restricted our attention to Hausdorff ample groupoids as this is the condition that guarantees that the convolution sum is finite [18, Propostion 3.5] and hence allows us to form this algebra.

This algebra was first introduced by Steinberg in [18], hence we refer to these algebras as *Steinberg algebras.* These algebras were also studied in [9] as generalizations of Leavitt path algebras. In that paper they prove that $A_{\mathbb{C}}(G_E)$ is isomorphic to the Leavitt path algebra $L_{\mathbb{C}}(E)$, where E is a directed row-finite graph with no sinks, and G_E its associated groupoid as in 2.7.

The aim of this thesis is to identify ideals in $A_R(G)$. As in the case for Leavitt path algbras we hope to show that the ideal structure of the algebra is determined entirely by the structure of the groupoid G. However when R is not a field we run into the problem of ideals in our ring generating ideals in the algebra, the following example demonstrates this.

Example 2.8. Let G be the trivial group consisting of just the identity e, and R a commutative ring with identity. Then $A_R(G)$ consists of the constant functions from e to the ring, hence we can see that $A_R(G) \cong R$. Therefore every ideal of the ring R is going to carried to a unique ideal of the algebra $A_R(G)$.

We can clearly see in this case that different rings will give the algebra different ideal structures. To get around this problem we introduce the idea of a basic ideal. These were first introduced by Tomforde in [20] for Leavitt path algebras who encountered a similar problem. We generalized the idea of basic ideals to Steinberg algebras in [8] with the following definition.

Definition 2.9. We call I a basic ideal of $A_R(G)$ if I is an ideal and if for any $r \in R$ and $K \subseteq G^{(0)}$ a compact open set, we have $r1_K \in I$ implies that $1_K \in I$. We say $A_R(G)$ is basically simple if it contains no non-trivial basic ideals.

We can see in our above example that every non-empty basic ideal must contain the function 1_e. Therefore as $R1_e = A_R(G)$ we can see that this algebra is basically simple. This agrees with the results of [8] which states that $A_R(G)$ is basically simple if and only if G is both effective and minimal. Note that G is effective in this case as $G \setminus G^{(0)}$ is empty.

3. Basic Results for Steinberg Algebras

In this section we provide the reader with some of the basic theory of Steinberg algebras that we will use in the rest of this thesis. This will cover many standard techniques currently used in the field as well an important theorem that will be important when classifying the ideal structure of these algebras. We begin this section by giving a second characterization of the algebra.

Remark 3.1. It is shown in [9, Lemma 3.3] that $A_R(G)$ can be realized as the finite linear R-span of characteristic functions 1_B where B is a compact open bisection. More specifically as shown in [9, Remark 2.4] we can write each $f \in A_R(G)$ as $f = \sum_{U \in F} a_U 1_U$ where F is a finite family of disjoint compact open bisections. Steinberg shows the multiplication of these characteristic functions is nicely behaved in [18, Proposition 3.5] with

$$1_B * 1_D = 1_{BD},$$

for any compact open bisections B and D.

This alternative way to view functions in $A_R(G)$ tends to be much easier to work with. This can be seen in the literature as most proofs in use this characterization. Therefore for the rest of this thesis we will use this important property without reference. In a slight abuse of notation we will often write $f \in A_R(G)$ as $f = \sum a_U 1_U$, whenever the reader encounters this always assume that this sum is finite with each U disjoint from the rest.

While multiplication between elements of $A_R(G)$ is usually quite complicated due to the convolution there are a few cases where it reduces to a much simpler form. One of these cases we make use of in this thesis is multiplication by characteristic functions supported entirely on units.

Remark 3.2. Given $K \subseteq G^{(0)}$ and $f = \sum a_U 1_U \in A_R(G)$ we have

$$(f * 1_K) = \sum a_U 1_{UK} = \sum a_U 1_{U \cap s^{-1}(K)} = f|_{G_K}$$

and similarly for right multiplication

$$(1_K * f) = \sum a_U 1_{KU} = \sum a_U 1_{U \cap r^{-1}(K)} = f|_{G^K}.$$

In particular this implies that multiplication between two characteristic functions supported entirely on units is pointwise, and hence multiplication between any two functions supported entirely on units is point-wise.

This is a good example of how this alternative characterization of functions can result in simpler methods. If one attempts to prove the above result with the initial classification of functions in $A_R(G)$ one can see it requires several more steps.

Due to the fact the functions of $A_R(G)$ are locally constant it may initially appear to the reader that these functions are not continuous. However in [8, Remark 2.2] it is shown that a locally constant function $f : G \to R$ is necessarily continuous for any topology on the ring R as long as G is Hausdorff, hence we can regard $A_R(G)$ as an algebra continuous functions. The continuity of these functions allows for some elegant proofs of results that initially do not seem straightforward (for example see Lemma 4.2).

It was shown in [9, Lemma 3.2] that the support of a continuous locally constant function $f : G \to \mathbb{C}$ is clopen. The same argument holds for functions $f : G \to R$ for any ring R, hence we can regard the support of a function $f \in A_R(G)$ as

$$\operatorname{supp}(f) := \{\gamma \in G : f(\gamma) \neq 0\}.$$

In the following sections we will often need to work with sub-algebras of the form $\{f \in A_R(G) : \operatorname{supp}(f) \subseteq H\}$ where H is an open invariant subset of $G^{(0)}$. We introduce a few alternative characterizations for this sub-algebra to simplify our notation.

We first note that $\{f \in A_R(G) : \operatorname{supp}(f) \subseteq H\} = \{f \in A_R(G) : f|_{G \setminus H} \equiv 0\}$. We give the quick proof of this below. Let $f \in \{f \in A_R(G) : \operatorname{supp}(f) \subseteq H\}$ and take $\alpha \in G \setminus H$, then $\alpha \notin H$ so $f(\alpha) = 0$. Let $f \in \{f \in A_R(G) : f|_{G \setminus H} \equiv 0\}$ and take $\alpha \notin H$, then $\alpha \in G \setminus H$ so $f(\alpha) = 0$.

We present a slightly more complicated characterization in the following remark.

Remark 3.3. For any open groupoid $H \subseteq G$ we can identify the sub-algebra $\{f \in A_R(G) : \operatorname{supp}(f) \subseteq H\}$ as the algebra $A_R(H)$. To do this we embed the elements of $A_R(H)$ into $A_R(G)$ with the inclusion map

$$\iota(f)(\gamma) = \begin{cases} f(\gamma) & : \gamma \in H \\ 0 & : \text{ otherwise} \end{cases}$$

and claim that this map is a homomorphism with $\iota(A_R(H)) = \{f \in A_R(G) : \operatorname{supp}(f) \subseteq H\}$.

Let $\iota(f) \in \iota(A_R(H))$, as H is open, open sets in H are open in G, so $\iota(f)$ is locally constant on G. We also have $\operatorname{supp}(\iota(f))$ is compact in G as for any open cover in G we can take the intersection with H to get an open cover in H, hence $\iota(f) \in A_R(G)$. We have $\operatorname{supp}(\iota(f)) \subseteq H$ by the definition of ι.

Let $f \in \{f \in A_R(G) : \operatorname{supp}(f) \subseteq H\}$, then by restricting the domain of f to H we can see that $f = \iota(f_H)$.

To complete the proof we need to show it is also a homomorphism, the only non-trivial requirement to check is to show it preserves algebra multiplication. Let $f, g \in A_R(H)$ and take $\gamma \in G$, if $\gamma \notin H$ then we are done. Otherwise

$$\iota(f*g)(\gamma) = (f*g)(\gamma) = \sum_{r(\beta)=r(\gamma)} f(\beta)g(\beta^{-1}\gamma) = \sum_{r(\beta)=r(\gamma)} \iota(f)(\beta)\iota(g)(\beta^{-1}\gamma) = (\iota(f)*\iota(g))(\gamma).$$

We simply refer to this sub-algebra with the slightly lazy notation $A_R(H)$. When the reader encounters this we mean the sub-algebra $\iota(A_R(H))$ of $A_R(G)$. It is important to note however that when we are dealing with a closed sub-groupoid H that we can not embed $A_R(H)$ in the same way. When we are dealing with these we regard $A_R(H)$ as an algebra in its own right.

An important special case that we will make use of is when E is an open subset of units, in which case $A_R(E)$ is an algebra where the convolution reduces to point-wise multiplication.

An important theorem for Steinberg algebras that we will use in the following sections is the Cuntz-Krieger uniqueness theorem [8, Theorem 3.2]. This theorem states that for an effective groupoid G, every non-injective R-algebra homomorphism from $A_R(G) \to A$ is non-zero on some $r1_K \in A_R(G^{(0)})$. This theorem is of use to us as it has an important corollary. This corollary states that for an effective groupoid, every non-trivial ideal contains a function supported entirely on the unit space of the groupoid. This corollary can in fact be expanded on as we do in Lemma 5.5.

4. Basic Ideal Structure

In this section we present our first classification of the basic ideal structure of Steinberg algebras. The main result of this section is based on [6, Theorem 5.9] which proves an analogous result for groupoid C^*-algebras. However the assumptions they use are weaker than the ones we use here, in particular they state that G must be second countable and topologically principle.

Theorem 4.1. *Let G be an ample, Hausdorff groupoid. Suppose G is strongly effective, then*

$$D \mapsto \{f \in A_R(G) : f|_{G_D} \equiv 0\}$$

is a bijection from the closed invariant subsets of $G^{(0)}$ onto the basic ideals of $A_R(G)$.

Before we can prove this theorem we first need to prove a few technical lemmas. The first two of these deal with the injectivity of the map.

Lemma 4.2. *Let G be an ample, Hausdorff groupoid, and D a subset of $G^{(0)}$. Then*

$$\{f \in A_R(G) : f|_{G_D} \equiv 0\} = \{f \in A_R(G) : f|_{G_{\bar{D}}} \equiv 0\}$$

Proof. Let $g \in \{f \in A_R(G) : f|_{G_D} \equiv 0\}$, then

$$\begin{aligned} g(s^{-1}(\bar{D})) &= g(\overline{s^{-1}(D)}) \text{ (as the source map is a continuous open map)} \\ &\subseteq \overline{g(s^{-1}(D))} \text{ (as } g \text{ is continuous)} \\ &= \overline{\{0\}} = \{0\}. \end{aligned}$$

Thus $g \in \{f \in A_R(G) : f|_{G_{\bar{D}}} \equiv 0\}$. The reverse inclusion is trivial as $G_D \subseteq G_{\overline{D}}$. □

Lemma 4.3. *Let G be an ample, Hausdorff groupoid, and D, E subsets of $G^{(0)}$. Then $\bar{D} = \bar{E}$ if and only if*

$$\{f \in A_R(G) : f|_{G_D} \equiv 0\} = \{f \in A_R(G) : f|_{G_E} \equiv 0\}.$$

Proof. Suppose that $\bar{D} = \bar{E}$, and let $g \in \{f \in A_R(G) : f|_{G_D} \equiv 0\}$. By Lemma 4.2 we have that $g \in \{f \in A_R(G) : f|_{G_{\bar{D}}} \equiv 0\} = \{f \in A_R(G) : f|_{G_{\bar{E}}} \equiv 0\}$ which implies that $g \in \{f \in A_R(G) : f|_{G_E} \equiv 0\}$ using Lemma 4.2 again. The reverse inclusion uses the same argument with E and D interchanged.

Now suppose that $\{f \in A_R(G) : f|_{G_D} \equiv 0\} = \{f \in A_R(G) : f|_{G_E} \equiv 0\}$, and let $\alpha \in \bar{D}$. Assume by way of contradiction that $\alpha \notin \bar{E}$, then there exists an open set U containing α that doesn't intersect E (we can assume $U \subseteq G^{(0)}$ as $G^{(0)}$ is open). As G is ample, there exists a compact open set $K \subseteq U$ containing α. We can see that K is disjoint from E, so by Lemma 4.2 we get

$$1_K \in \{f \in A_R(G) : f|_{G_E} \equiv 0\} = \{f \in A_R(G) : f|_{G_D} \equiv 0\} = \{f \in A_R(G) : f|_{G_{\bar{D}}} \equiv 0\}.$$

This implies that K and $\bar{D}$ are disjoint, a contradiction as α is contained in both sets. Thus $\bar{D} \subseteq \bar{E}$, the reverse inclusion follows with D and E interchanged. ☐

With these lemmas it is now easy to show the injectivity of the map defined in 4.1. We present this in a separate lemma as it does not require the assumption that G is strongly effective.

Lemma 4.4. *Let G be an ample Hausdorff groupoid and R a commutative ring with identity, then the map given in Theorem 4.1 is injective and maps into the basic ideals of $A_R(G)$.*

Proof. First we claim that if D is a closed invariant subset of $G^{(0)}$ then $\{f \in A_R(G) : f|_{G_D} \equiv 0\}$ is a basic ideal. To prove this let $g \in A_R(G)$, $h \in \{f \in A_R(G) : f|_{G_D} \equiv 0\}$ and $\alpha \in G_D$, then we have that

$$(g * h)(\alpha) = \sum_{\beta : r(\beta) = r(\alpha)} g(\beta) h(\beta^{-1}\alpha).$$

For each $\beta \in G$ we have that $s(\beta^{-1}\alpha) = \alpha^{-1}\beta\beta^{-1}\alpha = s(\alpha) \in D$. Thus $\beta^{-1}\alpha \in G_D$, so $h(\beta^{-1}\alpha) = 0$ and therefore $(g * h)(\alpha) = 0$. We also have that

$$(h * g)(\alpha) = \sum_{\beta : r(\beta) = r(\alpha)} h(\beta) g(\beta^{-1}\alpha).$$

As $\alpha \in G_D$ we get that $\alpha \in G^D$ as D is invariant, so $r(\alpha) \in D$. Thus for any $\beta \in G$ with $r(\beta) = r(\alpha)$ we get that $r(\beta) \in D$. Therefore $\beta \in G^D = G_D$ as D is invariant, so $h(\beta) = 0$. Hence $(h * g)(\alpha) = 0$, and so $\{f \in A_R(G) : f|_{G_D} \equiv 0\}$ is an ideal. The fact it is a basic ideal follows from the construction of the set, as for any $r \in R \setminus \{0\}$ and compact open $K \subseteq G^{(0)}$ we have that $r1_K|_{G_D} = 0 \Leftrightarrow 1_K|_{G_D} = 0$.

To see this map in injective, let D and E be closed subsets of $G^{(0)}$ such that $D \neq E$. Then Lemma 4.3 implies that $\{f \in A_R(G) : f|_{G_D} \equiv 0\} \neq \{f \in A_R(G) : f|_{G_E} \equiv 0\}$. ☐

To show the surjectivity we need to find an inverse map. We will show that given a basic ideal I, the map $I \mapsto \bigcap_{f \in I \cap A_R(G^{(0)})} f^{-1}(0)$ is the inverse.

Lemma 4.5. *Let G be an ample, Hausdorff groupoid, and I a basic ideal of $A_R(G)$. Then the set*

$$D := \bigcap_{f \in I \cap A_R(G^{(0)})} f^{-1}(0)$$

is a closed invariant subset of $G^{(0)}$.

Proof. If $I \cap A_R(G^{(0)})$ is empty, then $D = G^{(0)}$ which is trivially invariant, so we can assume $I \cap A_R(G^{(0)})$ is non-empty. We will show D is invariant by proving its complement is invariant. Suppose that $s(\gamma) \notin D$, then there exists $f \in I \cap A_R(G^{(0)})$ such that $f(s(\gamma)) \neq 0$. We can write

$$f = \sum_{K \in F} a_K 1_K,$$

where F is a finite collection of disjoint compact open subsets of $G^{(0)}$. We have that $s(\gamma) \in \text{supp}(f)$ so there exists a unique $K_0 \in F$ such that $s(\gamma) \in K_0$, and as G is ample there exists a compact open bisection B that contains γ. Note that we can assume that $s(B) \subseteq K_0$ by taking the intersection of B with $s^{-1}(K_0)$. Then

$$1_B * f * 1_{B^{-1}} = \sum_{K \in F} a_K 1_{BKB^{-1}} = a_{K_0} 1_{BK_0B^{-1}},$$

as $BKB^{-1} = \{r(\beta) : \beta \in B \text{ and } s(\beta) \in K\}$ for each $K \in F$, so we can see that $BKB^{-1} = \emptyset$ for every $K \neq K_0$ as F is a disjoint family. Thus $1_B * f * 1_{B^{-1}}(r(\gamma)) \neq 0$, so $r(\gamma) \notin D$. To see that D is closed notice that $f^{-1}(0)$ is closed for each $f \in I \cap A_R(G^{(0)})$ as f is continuous. Thus D is an intersection of closed sets, so it is closed. ☐

Before the reader carries on we recommend they review Remark 3.3 to understand the notation used in the following proofs.

Lemma 4.6. *Let G be an ample, Hausdorff groupoid, and I a basic ideal of $A_R(G)$. Then*

$$A_R(G^{(0)} \setminus D) = I \cap A_R(G^{(0)})$$

where

$$D := \bigcap_{f \in I \cap A_R(G^{(0)})} f^{-1}(0)$$

.

Proof. To show the reverse containment let $f \in I \cap A_R(G^{(0)})$, then $f(D) \equiv 0$ by the construction of D, thus $f \in A_R(G^{(0)} \setminus D)$. Now for the forward containment let $f \in A_R(G^{(0)} \setminus D)$, then for each $\mu \in \text{supp}(f)$, there exists a $f_\mu \in I \cap A_R(G^{(0)})$ and $r \in R \setminus \{0\}$ such that $f_\mu(\mu) = r$. We can write $f_\mu = \sum_{K \in F} a_K 1_K$ where F is a finite family of disjoint open subsets of $G^{(0)}$, hence there exists $K_\mu \in F$ such that $a_{K_\mu} = r$ and $\mu \in K_\mu$. As multiplication in $A_R(G^{(0)})$ is pointwise we can see that $r1_{K_\mu} = f_\mu * 1_{K_\mu} \in I \cap A_R(G^{(0)}$, therefore $1_{K_\mu} \in I \cap A_R(G^{(0)})$ as I is a basic ideal. As the collection $\{1_{K_\mu} : \mu \in \text{supp}(f)\}$ covers $\text{supp}(f)$ there exists a finite sub-collection Φ (of size n) that covers $\text{supp}(f)$. We will now construct $1_{\text{supp}(f)}$ from the elements of Φ. To do this first notice that for any

$U_1, U_2 \in \Phi$ that $1_{U_1 \cap U_2} = 1_{U_1} * 1_{U_2} \in I \cap A_R(G^{(0)})$. By induction this holds for any arbitrary intersections of elements of Φ. Thus using inclusion-exclusion we get

$$1_{\mathrm{supp}(f)} = \sum_{\Psi \subseteq \Phi} (-1)^{(|\Psi|+1)} 1_{(\bigcap \Psi)}$$

Thus $1_{\mathrm{supp}(f)} \in I \cap A_R(G^{(0)})$, and so $f = f * 1_{\mathrm{supp}(f)} \in I \cap A_R(G^{(0)})$. □

With this Lemma we can reprove one direction of the basic simplicity theorem in [8, Theorem 4.1] using a different argument. The authors of [8] prove this theorem using several technical lemmas. We present a much more straightforward approach that is easier for the reader to understand. The basic outline of the proof is to show that for an effective groupoid, every non-trivial ideal must contain a function entirely supported on the unit-space, and for a minimal groupoid, no such ideal can contain one. Note that in the original paper this is a if and only if statement.

Corollary 4.7. *Let G be an ample, Hausdorff groupoid. Suppose G is effective and minimal, then $A_R(G)$ is basically simple.*

Proof. Let I be a non-trivial basic ideal of $A_R(G)$, then by Lemma 4.5 we have that

$$D := \bigcap_{f \in I \cap A_R(G^{(0)}} f^{-1}(0)$$

is an invariant subset of $G^{(0)}$. As G is minimal we must have that either $D = G^{(0)}$ or $D = \emptyset$.

Suppose by way of contradiction that $D = G^{(0)}$, then by Lemma 4.6 we get that $I \cap A_R(G^{(0)}) = \{0\}$. However this is a contradiction as G is effective, so [8, Corollary 3.3] implies there exists $K \subseteq G^{(0)}$ such that $1_K \in I$. Therefore $D = \emptyset$, so Lemma 4.6 gives us that $I \cap A_R(G^{(0)}) = A_R(G^{(0)})$, hence $A_R(G^{(0)}) \subseteq I$. We next claim that $I = A_R(G)$, to see this let $f \in A_R(G)$ and write $f = \sum_{U \in F} a_U 1_U$, where F is a finite family of disjoint open bisections. Fix a $U \in F$, then $1_{s(H)} \in I$ which implies that $1_U = 1_U * 1_{s(U)} \in I$. Therefore $f = \sum_{U \in F} a_U 1_U \in I$. □

Remark 4.8. From this corollary we can get a result for finite group algebras over a field. As in a group the range and source maps of a group are equal so every non-trivial group is not effective. Therefore every non-trivial finite group algebra is non-simple.

With these lemmas in place we can now prove our main theorem in this section.

Proof of Theorem 4.1. By Lemma 4.4 is suffices to show that the map in surjective. Let I be a basic ideal of $A_R(G)$, then we claim that

$$\{f \in A_R(G) : f|_{G_{G^{(0)} \setminus D}} \equiv 0\} = I$$

with

$$D := \bigcap_{f \in I \cap A_R(G^{(0)})} f^{-1}(0)$$

which is a closed invariant set by Lemma 4.5.

We define the restriction from $A_R(G) \to A_R(G_D)$ with the map

$$\rho(f) = f|_{G_D}.$$

We claim this map is a homomorphism, the fact it preserves addition and ring multiplication is easy to verify. To see it preserves algebra multiplication let $f, g \in A_R(G)$, then for $\alpha \in G$

$$\rho(f * g)(\alpha) = \sum_{r(\beta)=r(\alpha)} f(\beta)|_{G_D} g(\beta^{-1}\alpha)|_{G_D} = \rho(f) * \rho(g).$$

Thus this map carries the ideal I of $A_R(G)$ to the ideal $\rho(I)$ of $A_R(G_D)$. We claim that $A_R(D)$ and the ideal $\rho(I)$ have trivial intersection. To see this let $1_K \in \rho(I)$ with $K \subseteq D$, then there exists $f \in I$ such that $1_K = \rho(f)$. Define $H := \text{supp}(f) \setminus G^{(0)}$, we can see H is compact and open as $G^{(0)}$ is closed, thus $s(H)$ is compact and open as s is a continuous open map. As $\rho(f) = f|_{G_D} = 1_K$ we can see that $s(H)$ and D are disjoint, to see this let $\mu \in s(H) \cap D$. As $\mu \in s(H)$ there exists $\gamma \in H$ with $s(\gamma) = \mu \in D$, hence $f(\gamma) \neq 0$. This however is a contraction as $\gamma \in G_D$ so $f(\gamma) = 1_K(\gamma) = 0$.

We claim that $(f - f * 1_{s(H)}) \in A_R(G^{(0)})$ and that $K \subseteq \text{supp}(f - f * 1_{s(H)})$. First let $\alpha \in G \setminus G^{(0)}$, we have $(f - f * 1_{s(H)})(\alpha) = f(\alpha) - f(\alpha)1_{s(H)}(s(\alpha))$. If $\alpha \notin \text{supp}(f)$ then this equals 0 and we are done, otherwise $\alpha \in H$ which implies $1_{s(H)}(s(\alpha)) = 1$. Either way $(f - f * 1_{s(H)})(\alpha) = 0$.

Now let $\mu \in K$, as D is disjoint from $s(H)$ we also have K is disjoint from $s(H)$. Therefore $1_{s(H)}(\mu) = 0$ and thus $(f - f * 1_{s(H)})(\mu) = f(\mu) - f(\mu)1_{s(H)}(\mu) = f(\mu)$. The fact $\rho(f) = 1_K$ completes the claim.

With these claims in place we can see that $1_K = (f - f * 1_{s(H)}) * 1_K \in I$, we now show that $K = \emptyset$. By way of contradiction assume that $K \neq \emptyset$, then as $K \subseteq D$ we get that $1_K \notin A_R(G^{(0)} \setminus D) = I \cap A_R(G^{(0)})$ by Lemma 4.6 giving the contradiction. Thus $K = \emptyset$ proving our claim that $A_R(D) \cap \rho(I) = \{0\}$. By assumption G_D is effective, so we can apply [8, Corollary 3.3] to see that every non-trivial basic ideal of $A_R(G_D)$ contains a function supported entirely on D, hence $\rho(I)$ must be trivial, that is $\rho(I) = \{0\}$

Let $f \in I$, then $\rho(f) = 0$ which implies $f \in A_R(G_{G^{(0)} \setminus D})$. For the reverse inclusion note that

$$\begin{aligned} A_R(G_{G^{(0)} \setminus D}) &= span(1_U : U \text{ is a compact open bisection such that } s(U) \subseteq G^{(0)} \setminus D) \\ &= span(1_U * 1_{s(U)} : U \text{ is a compact open bisection such that } s(U) \subseteq G^{(0)} \setminus D) \\ &\subseteq I, \end{aligned}$$

as $s(U) \subseteq G^{(0)} \setminus D$ implies $1_{s(U)} \in I$ by Lemma 4.6. $\square$

5. Non-Basic Ideals in Steinberg algebras

We now have a complete classification of the basic ideal structure of the Steinberg algebra of a strongly effective groupoid. In this section we attempt to extend this classification to include non-basic ideals of the algebra as well. A similar classification was done in [7, Section 6] for Kumjian-Pask algebras where they show that an ideal of the ring multiplied by the whole algebra can give non-basic ideals. In particular they prove that under the equivalent higher-rank graph conditions for effective and minimal that this map is surjective [7, Proposition 6.4]. In the more general case however, their map does not appear to include all the non-basic ideals of the algebra. We generalize this map to the Steinberg algebra setting as well as extending it so that it includes more non-basic ideals of the algebra. To show our map extends and generalizes the Kumjian-Pask version we prove that when G is effective and minimal the map is a bijection.

Theorem 5.1. *Let G be an ample, Hausdorff groupoid, and R a commutative ring with identity. Let H be the set of all open invariant subsets of $G^{(0)}$, and let P be the set of all functions $\pi : H \to \mathcal{L}(R)$ such that the following conditions hold:*

(1) *For every $E_1, E_2 \in H$*

$$E_1 \subseteq E_2 \implies \pi(E_2) \subseteq \pi(E_1).$$

(2) *For every $E \in H$, whenever $\phi \subseteq H$ such that $\bigcup \phi = E$ we have*

$$\pi(E) = \bigcap_{E_0 \in \phi} \pi(E_0)$$

Then the map $\Gamma : P \to \mathcal{L}(A_R(G))$ given by

$$\Gamma(\pi) = \operatorname{span}_{E \in H}\{\pi(E) A_R(G_E)\}$$

is an injection.

Before we begin with the proofs of this section we first introduce some remarks that allow us to more easily work with functions belonging to $\Gamma(\pi)$. As it stands now we can only regard functions of $\Gamma(\pi)$ as a sum of functions each multiplied by an element of R. This characterization is particularly difficult to work with so we introduce the following remarks to better understand these functions. The first of these gives us a more convenient way to write these functions.

Remark 5.2. Let $f = \sum_{E \in H} r_E \sum_i a_{U_E^i} 1_{U_E^i} = \sum_{E \in H, i} r_E a_{U_E} 1_{U_E} \in \Gamma(\pi)$, where each $r_E \in \pi(E)$ and each $s(U_E^i) \subseteq E$. It follows that for each E and i, $[s(U_E^i)] \subseteq E$, therefore using condition (1) of the map π we can assume that $E_U = [s(U)]$ as $r_E a_{U_E^i} \in \pi([s(U_E)])$. That is we can write $f \in \Gamma(\pi)$ as $f = \sum_{U \in F} a_U 1_U$ with each $a_U \in \pi([s(U)])$.

Next we note that as with general functions in the algebra $A_R(G)$ we can assume that functions in the set $\Gamma(\pi)$ are finite sums of disjoint compact open bisections. We do this using the same technique as in [9, Remark 2.5], however we include the whole argument for clarity as we can not assume the disjointification process gives that the ring scalars a_U are still elements of their associated $\pi([s(U)])$.

Remark 5.3. Let $f = \sum_{U \in F} a_U 1_U \in \Gamma(\pi)$, with each $a_U \in \pi([s(U)])$. We first construct the collection F^* which we define to be all possible intersections of elements of F. As $F \subseteq F^*$ this collection still covers $\text{supp}(f)$. As the elements of F are compact open bisections it follows that the elements of F^* are compact open bisections. We now disjointify this collection, for $U^* \in F^*$ we define $V_{U^*} := U^* \setminus \bigcup_{B^* \in F^* : U^* \not\subseteq B^*} B^*$. The collection $F_D^* := \{V_{U*} : U^* \in F*\}$ can easily be verified to be a disjoint cover of $\text{supp}(f)$ by compact open bisections.

With this new collection we can write $f = \sum_{V \in F_D^*} (\sum_{U \in F : V \subseteq U} a_U) 1_V$. Fix $V \in F_D^*$, for each $U \in F : V \subseteq U$ we have that $s(V) \subseteq s(U)$ and hence $[s(V)] \subseteq [s(U)]$, therefore we can use condition (1) of the map π to see that $a_U \in \pi([s(V)])$. This holds for every $U \in F : V \subseteq U$ so as $\pi([s(V)])$ is an ideal it follows that $\sum_{U \in F : V \subseteq U} a_U \in \pi([s(V)])$.

Putting these two remarks together gives us that we can write $f \in \Gamma(\pi)$ as $f = \sum a_U 1_U$ where each U is disjoint from the others and each $a_U \in \pi([s(U)])$.

With these two remarks in hand we can begin our proof of Theorem 5.1. However before we begin we first need the following technical lemma. The usefulness of our two previous remarks becomes apparent here as it makes the proof of this lemma almost trivial.

Lemma 5.4. *Let $\pi \in P$ with P defined as in Theorem 5.1. Suppose for some compact open $K \subseteq G^{(0)}$ and $r \in R$ that $r1_K \in \Gamma(\pi)$, then $r \in \pi([K])$.*

Proof. We can write $r1_K = \sum a_U 1_U$ where each $a_U \in \pi([s(U)])$, by Remark 5.3 we can assume each U is disjoint from the rest and hence each $U \subseteq K$ and $a_U = r$, in particular we get that $K = \bigcup U$. Thus

$$[K] = [\bigcup V] = r(s^{-1}(\bigcup V)) = \bigcup r(s^{-1}(V) = \bigcup [V],$$

so condition (2) of the map π implies that $\pi([K]) = \bigcap \pi([V])$. Putting this all together gives $r \in \pi([K])$. ◻

proof of Theorem 5.1. We first verify that this map gives us an ideal of $A_R(G)$. Fix $\pi \in P$ and let $f \in \Gamma(\pi)$. Using Remark 5.3 we can write $f = \sum a_U 1_U$ with each $a_U \in \pi([s(U)])$ and each U disjoint from the rest, this sum is finite so $f \in A_R(G)$.

To see $\Gamma(\pi)$ is closed under ring multiplication let $r \in R$, then

$$rf = r\sum a_U 1_U = \sum = ra_U 1_U \in \Gamma(\pi)$$

as each $\pi(E)$ is an ideal of R, so $ra_U \in \pi([s(U)])$.

To see $\Gamma(\pi)$ is closed under algebra multiplication by elements of $A_R(G)$ let $g \in A_R(G)$, then

$$f * g = \sum_{E\in H} a_U 1_U * g = \sum_{E\in H} a_U(1_U * g).$$

Fix such a U then by Lemma 4.4 $A_R(G_{[s(U)]})$ is an ideal of $A_R(G)$. Therefore $1_U * g \in A_R(G_{[s(U)]})$ and so $a_U 1_U * g \in \pi([s(U)])A_R(G_{[s(U)]})$.

We now show this map is injective, let $\pi_1, \pi_2 \in P$ such that $\Gamma(\pi_1) = \Gamma(\pi_2)$. We show that $\pi_1 = \pi_2$. To do this we fix an $E^* \in H$ and it suffices to show that $\pi_1(E^*) \subseteq \pi_2(E^*)$ as the reverse inclusion uses the same argument. Let $r \in \pi_1(E^*)$ and fix $\mu \in E^*$. As G is ample we can find a compact open neighborhood of μ, $K_\mu \subseteq E^*$. We have

$$\begin{aligned} r1_K &\in \Gamma(\pi_1) \text{ from condition (1) of the map } \pi \\ &= \Gamma(\pi_2). \end{aligned}$$

Thus by Lemma 5.4, $r \in \pi([K_\mu])$.

This holds for every $\mu \in E^*$, so as the collection of open invariant sets $\{[K_\mu] : \mu \in E^*\}$ covers E^* we can apply condition (2) of the map π to see

$$r \in \bigcap_{\mu\in E^*} \pi_2([K_\mu]) = \pi_2(E^*).$$

□

As the map Γ is an extension of the map introduced in [7, Section 6] for the Kumjian-Pask algebra setting we can recover their result generalized up to the Steinberg algebra setting. However to do this we need another corollary of the Cuntz-Krieger uniqueness theorem [8, Corollary 3.3]. If we look into the details of the proof of the corollary and its associated theorem given in [8, Theorem 3.2] we can see they actually prove this more general version but do not include it in their statement of the theorem. To save the reader working through the proof in [8] we include in here for completeness. We give a slightly different argument to avoid repetition.

Lemma 5.5. *Let G be an ample Hausdorff groupoid, and R a commutative ring with identity. Suppose G is effective and let I be an ideal of $A_R(G)$. Let $f = \sum_{U\in F} a_U 1_U \in I$, then for every $V \in F$ there exists non-empty compact open $K \subseteq s(V)$ such that $a_V 1_K \in I$.*

Proof. Fix f and V as in the statement of the proof. We have that $1_{V^{-1}} * f = a_V 1_{s(V)} + \sum_{U\in F\setminus\{V\}} a_U 1_{V^{-1}U} \in I$. We copy the technique we used in [8, Lemma 3.1] to show that $V^{-1}U$ is disjoint from $s(V)$ for each $U \in F \setminus \{V\}$. Fix such a U and let $\mu \in s(V)$, and suppose by way of contradiction that $\mu \in V^{-1}U$. Then there exists $\alpha \in V$ and $\beta \in U$ such that $\alpha^{-1}\beta = \mu$ which implies that $\beta = \alpha$, a contradiction as $V \cap U = \emptyset$.

Let $B \in F \setminus \{V\}$, as G is effective we can apply Lemma 2.4 to get non-empty compact open $K \subseteq s(V)$ such that $K(V^{-1}B)K = \emptyset$. Thus is follows from the fact that $V^{-1}B$ is disjoint from $s(V)$ that

$$\begin{aligned} 1_K * 1_{V^{-1}} * f * 1_K &= 1_K * a_V 1_{s(V)} * 1_K + \sum_{U\in F\setminus\{V\}} a_U 1_K * 1_{V^{-1}U} * 1_K \\ &= a_V 1_K + \sum_{U\in F\setminus\{B,V\}} a_U 1_{s^{-1}(K)\cap V^{-1}U\cap r^{-1}(K)} \in I. \end{aligned}$$

Note that now $s^{-1}(K) \cap V^{-1}U \cap r^{-1}(K) \subseteq V^{-1}U$ is disjoint from $K \subseteq s(V)$ for each $U \in F \setminus \{B, V\}$, so we can repeat this process until F is empty to get left with $a_V 1_K \in I$ for some non-empty compact open $K \subseteq s(V)$. ☐

Theorem 5.6. *Let G be an ample Hausdorff groupoid, and R a commutative ring with identity. Suppose G is effective and minimal, then the map Γ given in Theorem 5.1 is a bijection.*

Proof. We first note that in this case as G is minimal the map Γ reduces to

$$\pi \mapsto \pi(G^{(0)})A_R(G)$$

Using the result of Theorem 5.1 it suffices to show that this map is bijective. Let I be an ideal of $A_R(G)$, then we claim that $\pi(G) = \{r : r1_K \in I \text{ for all compact open } K \subseteq G^{(0)}\}$ satisfies $I = \pi(G)A_R(G)$. The fact that $\pi(G)$ is an ideal of R is easy to verify, and it trivially satisfies that necessary conditions as $G^{(0)}$ is the only non-trivial open invariant subset of $G^{(0)}$ from the minimality of G (this is also a consequence of Lemma 5.7).

Let $rf \in \pi(G)A_R(G)$, then $r1_K \in I$ for all compact open $K \subseteq G^{(0)}$, in particular $r1_{s(supp(f))} \in I$ and hence $rf = f * r1_{s(supp(f))} \in I$.

Let $f = \sum_{U\in F} a_U 1_U \in I$ and fix a $V \in F$. We can apply Lemma 5.5 to get a compact open $K \subseteq s(V)$ such that $a_V 1_K \in I$.

Now let $K' \subseteq G^{(0)}$ and take $\mu \in K'$. As G is minimal we must have that $[K] = G^{(0)}$ so there exists $\gamma_\mu \in G$ such that $r(\gamma_\mu) = \mu$ and $s(\gamma_\mu) \in K$. Let $B_\mu \subseteq G \setminus G^{(0)}$ be a compact open bisection containing γ_μ, then

$$1_{B_\mu} * a_V 1_K * 1_{B_\mu^{-1}} = a_V 1_{r(B_\mu \cap s^{-1}(K))} \in I.$$

As $\gamma_\mu \in B_\mu \cap s^-1(K)$ we get that $\mu = r(\gamma_\mu) \in r(B_\mu \cap s^-1(K))$.

Because of this we can see the collection $\{r(B_\mu \cap s^-1(K)) : \mu \in K'\}$ covers K', thus as K' is compact we can take a finite sub-collection which we will label $\{r(B_{\mu_t} \cap s^-1(K)) : 0 \leq t \leq M\}$. Note that these are compact open sets so by [9, Remark 24] we can assume they are disjoint, and hence

$$a_V 1_{K'} = \sum_{0 \leq t \leq M} a_V 1_{r(B_{\mu_t} \cap s^{-1}(K))} \in I.$$

Thus $a_V \in \pi(G)$ and therefore $a_V 1_V \in \pi(G^{(0)}) A_R(G)$. As V was chosen arbitrarily it follows that $f = \sum_{U \in F} a_U 1_U \in \pi(G) A_R(G)$. □

We would like to prove the map given in Theorem 5.1 is a bijection when G is strongly effective. Many examples suggests this conjecture is true, but unfortunately we have not been able to give a proof confirming our suspicions. We leave this as an open question and invite the reader to attempt to prove the result or construct a counter-example. Proving this result would be important as it has not been shown to be true for either Kumjian-Pask algebras, semi-group algebras, or Leavitt path algebras. To assist with any further research into the question we present below what we believe to be the inverse of the map Γ given in Theorem 5.1 and we show that is satisfies the required conditions.

Lemma 5.7. *Let the setting of this proof be the same as in Theorem 5.1. If I is an ideal of $A_R(G)$ then the map $\pi : H \to \mathcal{L}(A_R(G))$ defined by*

$$\pi(E) = \{r \in R : r1_K \in I \textit{ for all compact open } K \subseteq E\}$$

is in P.

Proof. To show $\pi \in P$ we must show it satisfies both the required conditions. We begin with condition (1), let $E_1, E_2 \in H$ such that $E_1 \subseteq E_2$. Suppose $r \in \pi(E_2)$, then $r1_{K_2} \in I$ for all compact open $K_2 \subseteq E_2$. Let $K_1 \subseteq E_1$ be compact open, then as $K_1 \subseteq E_1 \subseteq E_2$ we have $r1_{K_1} \in I$.

Now fix $E \in H$ and let $\phi \subseteq H$ such that $\bigcup \phi = E$, we want to show that $\pi(E) = \bigcup_{E^* \in \phi} \pi(E^*)$. Let $r \in \pi(E)$, then by condition (1) $r \in \pi(E^*)$ for all $E^* \subseteq E$. Hence $r \in \bigcup_{E^* \in \phi} \pi(E^*)$. Let $r \in \bigcup_{E^* \in \phi} \pi(E^*)$ and fix a compact open $K \subseteq E$. For each $\mu \in K$

there must exists some $K_\mu \in \phi$ that contains μ as ϕ covers E. Therefore as $r \in \pi(K_\mu)$ there exists a compact open $K_\mu \subseteq E_\mu$ that contains μ and with $r1_{K_\mu} \in I$. The collection $\{K_\mu : \mu \in K\}$ covers K as K is compact there exists a finite subcover, say $\{K_{\mu_i} : 0 \leq i \leq N\}$ for some N. We now construct the disjointification of this set as in [9, Remark 2.5] to get the collection $\{K^*_{\mu_i} : 0 \leq i \leq N\}$. By the nature of this construction each $K^*_{\mu_i} \subseteq K_{\mu_i}$ and so $r1_{K^*_{\mu_i}} = r1_{K^*_{\mu_i}} * 1_{K_{\mu_i}} \in I$, hence

$$r1_K = \sum_{0 \leq i \leq N} r1_{K^*_{\mu_i}} \in I.$$

□

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